

AD-A130 954

TWO DIMENSIONAL LINEAR PREDICTION MODELS PART 1  
SPECTRAL FACTORIZATION AN..(U) CALIFORNIA UNIV DAVIS  
SIGNAL AND IMAGE PROCESSING LAB S RANGANATH ET AL.  
MAY 83 SIPL-83-5 ARO-18532.3-EL

1/1

UNCLASSIFIED

F/G 12/1

NL

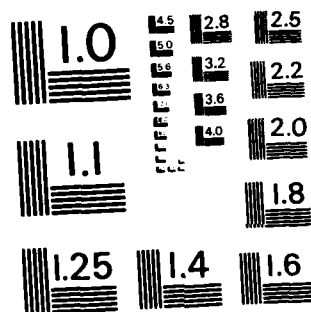
END

DATE

FILED

83

PT



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 18532.3-EL	2. GOVT ACCESSION NO. AD-A130 954	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Two Dimensional Linear Prediction Models. Part I: Spectral Factorization and Realization		5. TYPE OF REPORT & PERIOD COVERED Technical
6. PERFORMING ORG. REPORT NUMBER		8. CONTRACT OR GRANT NUMBER(s) DAAG29 82 K 0077
7. AUTHOR(s) Surendra Ranganath Anil K. Jain		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of California Davis, CA 95616		12. REPORT DATE May 83
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		13. NUMBER OF PAGES 55
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) JUL 29 1983
18. SUPPLEMENTARY NOTES The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) mathematical models algorithms predicting filters
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In this paper we present several results for three different canonical forms of linear prediction on a plane. These filters have causal, semicausal and noncausal prediction geometries. Starting from their properties we consider the problem of realization of these filters from a given power spectral density function (SDF). Since it is not possible in general to obtain rational spectral factors of a two dimensional SDF, we propose algorithms for obtaining rational approximations which are stable and converge to their limit (irrational) factors as the order of approximation is increased. It is also shown that the normal equations associated

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

83 07 28 030

ADA130954

DTIC FILE COPY

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

with the minimum variance two-dimensional prediction filters give a useful algorithm for obtaining rational approximations which are stable and converge to their unique limit filters. This result allows design of finite order, stable filters by solving a finite number of equations while realizing the given SDF arbitrarily closely.



SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

**UCD**

**University of California, Davis**



**TWO DIMENSIONAL LINEAR PREDICTION MODELS  
PART I: SPECTRAL FACTORIZATION AND REALIZATION**

by

**Surendra Ranganath  
Anil K. Jain**

**SIPL-83-5**

**May 1983**

Research supported in part by U.S. Army Research Office  
Grant DAAG29-82-K-0077 and in part by ONR Grant  
N00014-81-K-0191 under a SRO project.

TWO DIMENSIONAL LINEAR PREDICTION MODELS  
PART I: SPECTRAL FACTORIZATION AND REALIZATION  
by

Surendra Ranganath  
Anil K. Jain

SIPL-83-5

May 1983

Research supported in part by U.S. Army Research Office  
Grant DAAG29-82-K-0077 and in part by ONR Grant  
N00014-81-K-0191 under a SRO project.

TWO DIMENSIONAL LINEAR PREDICTION MODELS  
PART I: SPECTRAL FACTORIZATION AND REALIZATION

by

Surendra Ranganath\*

Anil K. Jain

Signal and Image Processing Laboratory  
Department of Electrical and Computer Engineering  
University of California  
Davis, California 95616

May 1983

Research supported in part by U.S. Army Research Office Grant DAAG29-82-K-0077  
and in part by ONR Grant N00014-81-K-0191 under a SRO project.

\*Surendra Ranganath is now with the Applied Research Group, Tektronix, Inc.,  
P.O. Box 500, M.S. 50-370, Beaverton, OR 97077.

# 2 ABSTRACT

In this paper we present several results for three different canonical forms of linear prediction on a plane. These filters have causal, semicausal and noncausal prediction geometries. Starting from their properties <sup>the given</sup> ~~we~~ consider the problem of realization of these filters from a given power spectral density function (SDF). Since it is not possible in general to obtain rational spectral factors of a two dimensional SDF, <sup>they</sup> ~~we~~ propose algorithms for obtaining rational approximations which are stable and converge to their limit (irrational) factors as the order of approximation is increased. It is also shown that the normal equations associated with the minimum variance two-dimensional prediction filters give a useful algorithm for obtaining rational approximations which are stable and converge to their unique limit filters. This result allows design of finite order, stable filters by solving a finite number of equations while realizing the given SDF arbitrarily closely.



## I. INTRODUCTION

Many digital signal processing algorithms are designed for the processing of random data. For instance, a digital image may be considered to be a sample function of a discrete, wide sense stationary random field, which is described in terms of its covariance function or equivalently the spectral density function (SDF). Although algorithms such as Wiener filtering [1], transform coding of images [2], etc. can be designed once the covariance function is given, it is sometimes useful to obtain a linear difference equation representation of the random field. The model, if sufficiently accurate, should realize the SDF of the random field closely. Such models are useful in image coding, recursive filtering, image synthesis, spectral estimation, etc.

These models can be realized by spectral factorization. In the one dimensional (1-D) case, this involves factoring the SDF of the random process into causal (recursive) and anticausal factors. The general techniques to do this include the Wiener-Doob method [3] and the linear prediction method [4]. The principle of the Wiener-Doob method is to map the poles and zeros of the SDF into singularities of the logarithm of the SDF. This allows easy decomposition of singularities into those that are within the unit circle, and into those that are outside, leading to minimum phase and maximum phase factors, respectively. The linear prediction method fits successively higher order autoregressive models to the given covariances by solving a finite set of Toeplitz equations. Under some mild conditions on the given SDF, the SDFs realized by the models can be shown to converge uniformly to the given SDF.

In two dimensions (2-D), spectral factorization is complicated by the lack of a fundamental theorem of algebra. Whittle [5] recognized this difficulty, and indicated a Wiener-Doob like technique for 2-D spectral factorization. This fact seems to have been ignored until it was rediscovered by Ekstrom and Woods [6].

They provided an extension of the 1-D Wiener-Doob principle to 2-D to obtain recursive filter designs from given frequency magnitude specifications. The resulting infinite order causal factors were approximated to finite order by using suitable truncation and windowing.

Subsequently, Marzetta [7,8] approached the spectral factorization problem by extending the results of 1-D prediction theory to 2-D. The recursive factors obtained here are of infinite order in at least one of the dimensions. Also, the exact solution can only be obtained by solving an infinite set of linear equations. A 2-D analog of Levinson's algorithm was devised to obtain the theoretically guaranteed solution. More importantly, it was shown that a one-to-one relationship exists between the reflection coefficients obtained in the Levinson algorithm, the associated prediction error filters, and the covariances of the random field used in the algorithm. Also, it was proven that reflection coefficients given on a finite support yield finite support causal factors. These facts were used to design an approximate spectral factorization algorithm where the reflection coefficients are sequentially chosen on a finite lattice to minimize a prediction error functional.

In [9], Jain studied a class of hyperbolic, parabolic, and elliptic partial differential equations, and from their finite difference approximations developed finite order causal, semicausal, and noncausal representation for 2-D random fields. In [10], higher order models were considered, and it was shown that the coefficients could be obtained by solving a finite set of block Toeplitz equations. Though a unique solution may be found for the model parameters, there is no one-to-one correspondence between the coefficients and the covariances used in model realization. Also, model stability is not guaranteed. The advantage, however, is that only a finite Toeplitz set of equations need be solved.

An alternate method of obtaining models by factorizing the SDF,  $S(z_1, z_2)$  is shown in [7,8] for the causal case and in [10] for both, causal and semicausal

cases. Here, a set of normal equations, parametric in  $z_1$  must be sequentially solved for larger and larger orders, resulting in factors which are irrational in  $z_1$ . Also in [10], the Wiener-Doob principle has been extended to obtain semicausal representations.

In this paper we present several results concerning realization of noncausal, semicausal and causal models by linear prediction principles. For each class of models we present two types of algorithms, A1 and A2. In algorithm A1 we start with the given power spectrum density function (SDF) to obtain infinite order (irrational) realizations. A rationalization procedure is incorporated into the algorithms to provide stable models. These algorithms will theoretically require, as in all prior methods, solution of an infinite number of equations. A Levinson type algorithm presented for semicausal models turns out to be an extension of a similar algorithm [7,8] for causal models. The results for noncausal models do not require spectral factorization.

The algorithms A2 yield approximate rational realizations of 2-D SDFs which are obtained by solving finite number of equations. Our results give asymptotic behavior of properties such as spectral match, stability, covariance match, etc., of these models as their order is increased to infinity. An important consequence is that one obtains a practical procedure for designing stable causal, semicausal and noncausal models which remain finite in order while matching the given spectra arbitrarily closely.

In Section II, we provide the necessary definitions and properties to aid in the understanding of the subsequent sections. One dimensional noncausal models are analyzed in Section III, since the considerations involved there lead to insight into 2-D model behaviour. In Section IV, we present results on realization of 2-D noncausal models, by solving finite order block Toeplitz equations. In Sections V and VI, results for semicausal and 2-D causal models are presented. In Section VII, we give examples verifying the theoretical results of the earlier sections.

## II. NOTATION, DEFINITION AND PROPERTIES

### 2.1 Notation

- a) Two dimensional (2-D) sequences are denoted by  $u(i,j)$ ,  $\epsilon(i,j)$ , etc; where  $(i,j)$  are integers defined on a regular 2-D lattice.
- b) Matrices are denoted by upper case letters, e.g.  $U$ ,  $A$ ,  $R$ , etc. For example,

$$U \triangleq \{u(i,j) ; 1 \leq i \leq N, 1 \leq j \leq M\}$$

is a  $N \times M$  matrix of elements  $u(i,j)$ . The  $j^{\text{th}}$  column of  $U$  is written as  $\underline{u}_j$  and the  $(i,j)^{\text{th}}$  element of  $U$  is written as  $u(i,j)$ .

- c) The transpose of  $U$  is denoted as  $U^T$ . The complex conjugate of  $U$  is denoted as  $U^*$ .
- d) The z-transform of a sequence  $u(i,j)$  is denoted as  $U(z_1, z_2)$ , and  $U(\omega_1, \omega_2)$  denotes the z-transform evaluated on the unit circles,  $|z_1| = |z_2| = 1$ .
- e) The maximum lower bound of a function  $f(x)$  on a domain  $X$  is denoted as  $\inf[f(x)]$ , and its minimum upper bound is denoted  $\sup[f(x)]$ .

### 2.2 Definitions and Properties

- 1) A discrete random field  $\{u(i,j)\}$  is a 2-D sequence of random variables  $u(i,j)$ . If the random variables are real and jointly Gaussian, then it is called a real, Gaussian random field.
- 2) The random field  $\{u(i,j)\}$  is called wide sense stationary (or just stationary) if its mean is constant, and its covariance is independent of spatial translation, i.e.

$$E[u(i,j)] = \mu(i,j) = \mu$$

$$\text{cov}[u(i,j)u(m,n)] \triangleq E[u(i,j) - \mu](u(m,n) - \mu) = r_u(i-m, j-n)$$

When there is no fear of confusion, the subscript  $u$  will be dropped

from  $r_u(\cdot, \cdot)$ . Henceforth, we will consider real, zero mean, stationary random fields.

- 3) The random field  $\{u(i, j)\}$  is called white if the random variables  $u(i, j)$  are mutually uncorrelated, i.e.

$$r_u(k, \ell) = \sigma^2 \delta(k, \ell)$$

where  $\sigma^2$  is the variance of the field and  $\delta(k, \ell)$  is the Kronecker delta function.

- 4) The spectral density function (SDF),  $S(z_1, z_2)$  of  $\{u(i, j)\}$  is called positive analytic (PA) [8] if  $S(\omega_1, \omega_2) > 0$  and if  $S(z_1, z_2)$  is analytic in a neighborhood of  $|z_1| = |z_2| = 1$ . Let this neighborhood be

$$\Gamma = \{\gamma_1 < |z_1| < 1/\gamma_1, \gamma_2 < |z_2| < 1/\gamma_2; 0 < \gamma_1, \gamma_2 < 1\} \quad (1)$$

Then,

- a)  $S(z_1, z_2)$  has a unique Laurent expansion [11] given by

$$S(z_1, z_2) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} r(k, \ell) z_1^{-k} z_2^{-\ell}, \quad z_1, z_2 \in \Gamma \quad (2)$$

- b) Since  $S(\omega_1, \omega_2) > 0$ , by continuity, there exists another neighborhood

$$\alpha = \{\alpha_1 < |z_1| < 1/\alpha_1, \alpha_2 < |z_2| < 1/\alpha_2; 0 < \alpha_1, \alpha_2 < 1\} \quad (3)$$

where  $S^{-1}(z_1, z_2)$  is PA, and has the Laurent expansion

$$S^{-1}(z_1, z_2) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} r^{-}(k, \ell) z_1^{-k} z_2^{-\ell}, \quad z_1, z_2 \in \alpha \quad (4)$$

- c) Convergence of the series in (2) and (4) implies that the sequences  $r(k, \ell)$  and  $r^{-}(k, \ell)$  are exponentially bounded:

$$\left. \begin{aligned} |r(k, \ell)| &< A \gamma_1^{|k|} \gamma_2^{|\ell|} \\ |r^{-}(k, \ell)| &< B \alpha_1^{|k|} \alpha_2^{|\ell|} \end{aligned} \right\} \quad \forall(k, \ell) \quad (5)$$

for some positive constants A, B.

d) The sequence of functions  $\{r_\ell(z_1)\}$  defined as

$$r_\ell(z_1) \triangleq \sum_{k=-\infty}^{\infty} r(k,\ell) z_1^{-k} = \frac{1}{2\pi j} \int_{|z_2|=1} z_2^{\ell-1} S(z_1, z_2) dz_2 \quad (6)$$

i) are analytic in the neighborhood  $\{\gamma_1 < |z_1| < 1/\gamma_1\}$

ii) form a positive definite sequence on  $|z_1| = 1$ , i.e; for any  $x_k(z_1)$  not identically zero,

$$\sum_{k,\ell} x_k(z_1) r_{k-\ell}(z_1) x_\ell^*(z_1) > 0, \quad |z_1| = 1$$

This means that the Toeplitz matrix

$$R_q(z_1) = \begin{bmatrix} r_0(z_1) & r_1(z_1^{-1}) & \dots & r_q(z_1^{-1}) \\ r_1(z_1) & & & \\ \vdots & & & \\ r_q(z_1) & & r_1(z_1) & r_0(z_1) \end{bmatrix} \quad (7)$$

is Hermitian positive definite on  $|z_1| = 1$ .

e) The sequence  $\{r(k,\ell)\}$  is also positive definite.

The following definitions and properties associated with linear prediction, which are discussed in greater detail in [10], will be needed here.

5) Let  $\bar{u}(i,j)$  denote a linear prediction (LP) estimate of the random variable  $u(i,j)$ . Then

$$\bar{u}(i,j) = \sum_{(m,n) \in \hat{S}} a(m,n) u(i-m, j-n) \quad (8)$$

The  $a(m,n)$  are called the predictor coefficients and  $\hat{S}$ , a subset of the 2-D lattice is called the prediction region.

The geometry of  $\hat{S}$  depends on the type of prediction estimate considered, viz; causal, semi-causal or non-causal. With a hypothetical scanning mechanism that scans sequentially from top to bottom and left to right, the three prediction regions may be defined as follows:

Causal Prediction:

$$\hat{S}_1 = (m,n): \{n \geq 1, \forall m\} \cup \{n=0, m \geq 1\} \quad (9)$$

Semi-causal Prediction:

$$\hat{S}_2 = (m,n): \{n \geq 1, \forall m\} \cup \{n=0, \forall m \neq 0\} \quad (10)$$

Non-causal Prediction:

$$\hat{S}_3 = (m,n): \{\forall (m,n) \neq (0,0)\} \quad (11)$$

Also, we define

$$S_i = \hat{S}_i \cup (0,0), \quad i=1,2,3 \quad (12)$$

The above prediction regions are depicted in Fig. 1. In practice, only a finite number of nearest neighbor samples from prediction windows  $\hat{W}_i \subset \hat{S}_i$  can be used in the prediction. Prediction coefficients  $\{a(m,n)\}$  defined on  $\hat{S}_i$  are said to have continuous support whereas if defined only on the  $\hat{W}_i$  they have discontinuous support.

The prediction windows  $\hat{W}_i$  that we consider have rectangular support, for convenience:

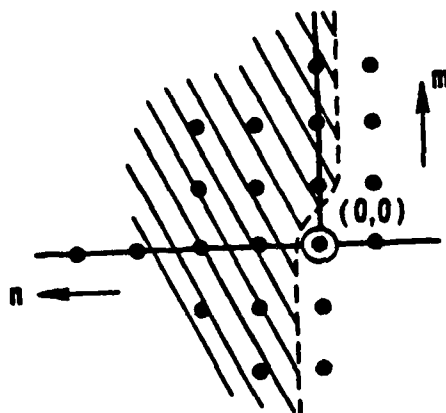
$$\hat{W}_1 = (m,n): \{1 \leq n \leq q, -p \leq m \leq p \text{ \& } n=0, 1 \leq m \leq p\} \quad \text{causal} \quad (13)$$

$$\hat{W}_2 = (m,n): \{0 \leq n \leq q, -p \leq m \leq p, (m,n) \neq (0,0)\} \quad \text{semi-causal} \quad (14)$$

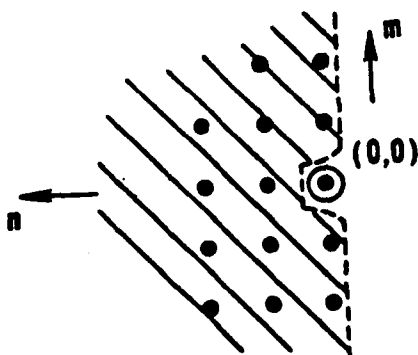
$$\hat{W}_3 = (m,n): \{-q \leq n \leq q, -p \leq m \leq p, (m,n) \neq (0,0)\} \quad \text{non-causal} \quad (15)$$

$$W_i = \hat{W}_i \cup (0,0), \quad i=1,2,3 \quad (16)$$

Causal Prediction:  $\hat{S}_1$



Semicausal Prediction:  $\hat{S}_2$



Noncausal Prediction:  $\hat{S}_3$

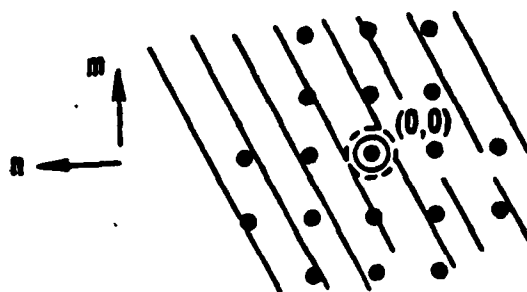


Fig. 1: Examples of Causal, Semicausal and Noncausal Prediction Regions



The prediction region  $\hat{S}_1$  in (9) is also called the non-symmetric half plane (NSHP) in the literature [6].

- 6) The predictor coefficients  $a(m,n)$  in (8) must be chosen according to some criterion. Minimum Variance Predictors are designed to minimize the variance of the prediction error, i.e.

$$\beta^2 = \min E[\varepsilon^2(i,j)], \quad \varepsilon(i,j) \triangleq u(i,j) - \bar{u}(i,j) \quad (17a)$$

For Gaussian random fields, the predictor will be linear. The orthogonality condition associated with minimum variance criterion is

$$E\{[u(i,j) - \sum_{(m,n) \in \hat{S}_x} a(m,n)u(i-m,j-n)]u(i-k,j-l)\} = \beta^2 \delta(k,l), \\ (k,l) \in S_x, \quad x=1,2,3$$

which gives

$$r(k,l) - \sum_{(m,n) \in \hat{S}_i} a(m,n)r(k-m,l-n) = \beta^2 \delta(k,l), \quad (k,l) \in S_i, \quad i=1,2,3 \quad (17b)$$

If the prediction window has finite support  $\hat{W}_i$ ,  $i=1,2,3$ , then the optimum predictor will satisfy (17b) where the summation is over  $(m,n) \in \hat{W}_i$ , i.e.,

$$r(k,l) - \sum_{m,n \in \hat{W}_i} a(m,n)r(k-m,l-n) = \beta^2 \delta(k,l), \quad (k,l) \in S_i, \quad i=1,2,3 \quad (17c)$$

Note that (17c) must be satisfied for  $\forall (k,l) \in S_i \Rightarrow W_i$  by the finite support optimum predictor.

- 7) A Stochastic Representation for a Gaussian field  $\{u(i,j)\}$  may be written as

$$u(i,j) = \bar{u}(i,j) + \varepsilon(i,j) \quad (18)$$

where  $\bar{u}(i,j)$  is an arbitrary prediction of  $u(i,j)$ , and  $\{\varepsilon(i,j)\}$  is another random field such that (18) realizes the covariance properties of  $\{u(i,j)\}$ . Here we consider minimum variance representations

(MVRs) where  $\bar{u}(i,j)$  is chosen to be a causal, semi-causal or non-causal minimum variance predictor. The difference in prediction geometries determine the properties of the prediction error field  $\{\epsilon(i,j)\}$ . As shown in [10],  $\{\epsilon(i,j)\}$  is

- a) a white noise field for causal MVRs.
- b) white in the causal dimension and correlated in the non-causal dimension for semi-causal MVRs.
- c) a correlated field for non-causal MVRs.

These properties are important because they point out that for semicausal and noncausal geometries white noise driven models will not have the minimum variance property.

- 8) A MVR as in (18) can be considered to be a linear difference equation with input  $\{\epsilon\}$  and output  $\{u\}$ . The SDF of  $\{u\}$  is then given by

$$S_u(z_1, z_2) = \frac{S_\epsilon(z_1, z_2)}{A(z_1, z_2)A(z_1^{-1}, z_2^{-1})} \quad (19)$$

$$\text{where } A(z_1, z_2) = 1 - \sum_{(m,n) \in \hat{S}_i} a(m,n) z_1^{-m} z_2^{-n} \quad (20)$$

is called the prediction error filter (PEF) for any prediction geometry.

- 9) The MVR of (18) is stable in the bounded input, bounded output (BIBO) sense if the PEF,  $A(z_1, z_2)$  satisfies certain conditions. For the three prediction windows in (13-15), the MVR is stable if and only if

$$a) \quad \left. \begin{array}{l} A(z_1, z_2) \neq 0, \quad |z_1| \geq 1, \quad z_2 = \infty \\ A(z_1, z_2) \neq 0, \quad |z_1| = 1, \quad |z_2| \geq 1 \end{array} \right\} \text{causal} \quad (21)$$

$$A(z_1, z_2) = 1 - \sum_{m=1}^p a(m,0) z_1^{-m} - \sum_{m=-p}^p \sum_{n=1}^q a(m,n) z_1^{-m} z_2^{-n}$$

$$b) \quad A(z_1, z_2) \neq 0, \quad |z_1| = 1, \quad |z_2| \geq 1 \quad \text{Semi-causal} \quad (22)$$

$$A(z_1, z_2) = 1 - \sum_{\substack{m=-p \\ m \neq 0}}^p a(m, 0) z_1^{-m} - \sum_{m=-p}^p \sum_{n=1}^q a(m, n) z_1^{-m} z_2^{-n}$$

$$c) \quad A(z_1, z_2) \neq 0, \quad |z_1| = 1, \quad |z_2| = 1 \quad \text{Non-causal} \quad (23)$$

$$A(z_1, z_2) = 1 - \sum_{m=-p}^p \sum_{\substack{n=-q \\ (m,n) \neq (0,0)}}^q a(m, n) z_1^{-m} z_2^{-n}$$

10) Closely related to stability is the concept of minimum phase. A stable causal filter is minimum phase if it has a causal stable inverse. A stable semi-causal filter is semi-minimum phase if it has a stable semi-causal inverse. Marzetta [7,8] also considers analytic minimum phase (AMP) filters i.e. minimum phase filters that are also analytic in a neighborhood of the unit circles.

### III. ONE DIMENSIONAL NON-CAUSAL MODELS

First we present some results on 1-D non-causal representations with regard to properties such as stability, correlation match, etc., to provide direct insight into similar questions for 2-D representations. The 1-D non-causal representations are important in their own right since they find application in data compression and restoration of images using image transforms (for example, see [13,14]). These results provide certain extensions of the theory of non-causal models developed in [10].

A non-causal MVR for a Gaussian random process  $\{u(k)\}$  may be written

$$u(k) = \bar{u}(k) + \varepsilon(k) \quad (24)$$

$$\bar{u}(k) = \sum_{\ell \neq 0} a(\ell) u(k-\ell) \quad (25)$$

where  $\bar{u}(k)$  is the minimum variance prediction estimate of  $u(k)$  based on all the past and future, and  $\{\epsilon(k)\}$  is the prediction error process. The associated orthogonality condition,  $E[\bar{u}(k)\epsilon(k)] = 0$ , implies the normal equations

$$r(n) = \sum_{\ell \neq 0} a(\ell) r(n-\ell) + \beta^2 \delta(n), \quad \forall n \quad (26)$$

where  $r(n)$  are the covariances of  $\{u(k)\}$  and  $\beta^2 = \min E[\epsilon^2(k)]$ . If  $S(z)$  is the SDF of  $\{u(k)\}$  then the above equation can be written as

$$S(z) = \beta^2 / A(z), \quad A(z) \triangleq 1 - \sum_{\ell \neq 0} a(\ell) z^{-\ell} \quad (27)$$

where  $A(z)$  is the PEF. The solution of (27) gives the parameters of the non-causal MVR as [10]

$$\left. \begin{aligned} a(\ell) &= a(-\ell) = -r^-(\ell) / r^-(0) \\ \beta^2 &= 1 / r^-(0) = 1 / \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} S^{-1}(\omega) d\omega \right] \end{aligned} \right\} \quad (28)$$

where  $r^-(\ell)$  are the Fourier series coefficients of  $S^{-1}(\omega)$ . Unlike the autoregressive (AR) models where  $\{\epsilon(k)\}$  turns out to be a white process, in the non-causal MVR it is non-white. This can be seen by multiplying (24) by  $\epsilon(k-n)$  and taking expectations, which yields

$$r_{\epsilon}(n) = \beta^2 \delta(n) - \beta^2 \sum_{\ell \neq 0} a(\ell) \delta(n-\ell) \quad \text{or} \quad S_{\epsilon}(z) = \beta^2 A(z) \quad (29)$$

Thus, the non-causal PEF is not a whitening filter for the process  $u(k)$ .

In (25), the prediction estimate  $\bar{u}(k)$  uses all the past and future samples of  $u(k)$  in the prediction. If however,  $\{u(k)\}$  is a  $p^{\text{th}}$  order Markov process [15], then

$$\bar{u}(k) | \{u(k-\ell), \ell \neq 0\} = \bar{u}(k) | \{u(k-\ell), 1 \leq |\ell| \leq p\}. \quad (30)$$

i.e. all the information needed for minimum variance non-causal prediction of

$u(k)$  is contained in the immediate  $p$  past and future samples of  $u(k)$ .

Thus, the MVR of this process is finite order with  $\{a(\ell)=0, |\ell|>p\}$ . The other parameters,  $\beta^2$ ,  $\{a(\ell), |\ell|\leq p\}$  can be obtained by solving a finite subset of (26) corresponding to  $|n|\leq p$ . This solution will also satisfy (26) for  $|n|>p$ . Eqn. (26) implies  $\{u\}$  and  $\{\epsilon\}$  are uncorrelated with cross covariance  $r_{u\epsilon}(k) = \beta^2\delta(k)$ , and (29) implies that  $\{\epsilon\}$  is a moving average process with

$$S_{\epsilon}(z) = \beta^2 A(z); A(z) = 1 - \sum_{\substack{\ell=-p \\ \ell \neq 0}}^p a(\ell)z^{-\ell} \quad (31)$$

Therefore, the SDF realized by this MVR is all pole and rational. Now, if the SDF of  $\{u\}$  is irrational (but well behaved), a finite order MVR will not exist for the process. In practice, therefore, one approximates the infinite MVR by a finite order minimum variance prediction model. The considerations involved in designing models can be examined in relation to the properties of finite order MVRs. If a  $p^{\text{th}}$  order minimum variance predictor,  $\bar{u}(k) = \sum_{\substack{\ell=-p \\ \ell \neq 0}}^p a_p(\ell)u(k-\ell)$  is used to predict  $u(k)$ , and  $\epsilon_p(k)$  is the prediction error, then one can write  $u(k) = \bar{u}(k) + \epsilon_p(k)$ . The associated orthogonality condition gives

$$r(n) = \sum_{\substack{\ell=-p \\ \ell \neq 0}}^p a_p(\ell)r(n-\ell) + \beta_p^2\delta(n), \quad |n| \leq p \quad (32)$$

where  $\beta_p^2 = \min E[\epsilon_p^2(k)]$

In matrix form, the above equation can be written as

$$R_{p-p} \underline{a}_p = \beta_{p-p}^2 \underline{1}_p \quad (33a)$$

where

$$\underline{a}_p = [-a_p(-p) \dots -a_p(-1) \quad 1 \quad -a_p(1) \dots -a_p(p)]^T \quad (33b)$$

$$\underline{1}_p = [0 \dots \dots \dots 0 \quad 1 \quad 0 \dots \dots \dots 0]^T \quad (33c)$$

$$R_p \triangleq \begin{bmatrix} r(0) & r(-1) & \dots & r(-2p) \\ r(1) & & & \\ \vdots & & & \\ r(2p) & & & \end{bmatrix} \quad (33d)$$

The solution of (33) yields the coefficients

$$\beta_p^2 = 1/[R_p^{-1}]_{p+1,p+1} \quad (34)$$

$$a_p(i) = -\beta_p^2 [R_p^{-1}]_{i,p+1} \quad (35)$$

If  $R_p$  is positive definite (as will be the case when  $S(\omega) > 0$ ), a unique solution exists. Now it can be proven that the SDF of the residual process  $\{\epsilon_p(k)\}$  is given by

$$S_{\epsilon_p}(z) = [\beta_p^2 + S_{\epsilon_p u}(z)]A_p(z) \quad (36)$$

where  $S_{\epsilon_p u}(z)$  is the cross SDF of  $\{\epsilon_p\}$  and  $\{u\}$ . In general, no closed form expression can be found for  $S_{\epsilon_p u}(z)$ , and the residual process  $\{\epsilon_p\}$  is not simply a moving average. Thus, the  $p^{\text{th}}$  order non-causal predictor equation realizes the SDF of  $\{u\}$  exactly as

$$S_u(z) = [\beta_p^2 + S_{\epsilon_p u}(z)]/A_p(z) \quad (37)$$

As a representation for  $\{u\}$ , the exact predictor equation is not useful since the forcing function  $\{\epsilon_p(k)\}$  is inadequately characterized. A more practical model for  $\{u\}$  is obtained by constructing a process  $\{\tilde{u}\}$  with finite order MVR

$$\tilde{u}(k) = \sum_{\substack{\ell=-p \\ \ell \neq 0}}^p a_p(\ell) \tilde{u}(k-\ell) + \tilde{\epsilon}(k) \quad (38)$$

where  $\{\tilde{e}\}$  is chosen to be a moving average with  $S_{\tilde{e}}(z) = \beta_p^2 A_p(z)$ . This MVR will realize the rational, all pole SDF  $S_{\tilde{u}}(z) = \beta_p^2 / A_p(z)$ , and can also be considered as a model for  $\{u\}$ . Leaving aside the important question of model accuracy (for the time being), we formally write the definition of a model, for noncausal MVRs.

Definition: A  $p^{\text{th}}$  order minimum variance non-causal model for a process  $\{u(k)\}$  is a finite order MVR for another process  $\{\tilde{u}(k)\}$ , which is realized by solving the normal equations (33) for  $\{a_p(l), \beta_p^2\}$  and forcing the representation (38) by  $\{\tilde{e}\}$  with  $S_{\tilde{e}}(z) = \beta_p^2 A_p(z)$ .

With models being formally defined, we state some properties of these.

Properties:

$$1) \quad a_p(i) = a_p(-i) \quad ; \quad i = 1, 2, \dots, p$$

This is so since  $R_p^{-1}$  is persymmetric (symmetric about the cross diagonal) and  $\underline{a}_p$  is proportional to the middle column of  $R_p^{-1}$ .

$$2) \quad \beta_p^2 \geq \beta_{p+1}^2$$

i.e.  $\{\beta_p^2\}$  is a monotonically non-increasing sequence with increasing order  $p$ . This is intuitively obvious since for stationary processes inclusion of more samples in the prediction cannot increase the prediction error.

3) If  $\{u\}$  is  $p^{\text{th}}$  order Markov, then the  $p^{\text{th}}$  order non-causal model is the non-causal MVR of the process. This can easily be seen by noting that the SDF of this Markov process is of the form  $S(z) = C/A_p(z)$  where  $C$  is a constant. The inverse transform of this equation is of the form of (26). The subset of this equation corresponding to  $|n| \leq p$  is exactly the realization equation of the  $p^{\text{th}}$  order model.

4) The definition of a  $p^{\text{th}}$  order non-causal model implies that the spectral estimate from the model can be taken as

$$\hat{S}_p(z) = \beta_p^2 / A_p(z) \quad (39)$$

This implies that the non-causal model will be stable if and only if  $A_p(z)$  has no roots on the unit circle. Unlike the causal (AR) model, where stability of the model is equivalent to the positive definiteness of the covariance sequence, a corresponding result has not yet been discovered for non-causal models.

- 5) The correlation matching property of AR models does not hold for these non-causal models. In fact, examination of (33) and Property 1) shows that  $2p$  covariances are used to calculate  $p$  model coefficients. Hence, given a set of admissible model coefficients, there exists an infinite number of covariance sequences that will satisfy (33). Also, realization of non-causal models does not require spectral factorization.

The questions regarding stability and correlation matching properties which are measures of the goodness of the model are answered formally in the following theorem.

Theorem 1: Let  $\{u(k)\}$  be a Gaussian random process with PA SDF  $S(z)$ . Then the following may be proven:

- a) Convergence of PEFs: The sequence of non-causal PEFs

$$A_p(z) = 1 - \sum_{\substack{\ell=-p \\ \ell \neq 0}}^p a_p(\ell) z^{-\ell}$$

converges uniformly in some neighborhood of  $|z| = 1$  to an analytic limit PEF

$$\lim_{p \rightarrow \infty} A_p(z) = A(z)$$

The sequence of prediction error variances converges to a positive lower limit,  $\beta^2$

$$\lim_{p \rightarrow \infty} \beta_p^2 = \beta^2 = 1/2\pi \int_{-\pi}^{\pi} S^{-1}(\omega) d\omega$$



- b) Stability: a  $p^{\text{th}}$  order non-causal model is stable if and only if

$$A_p(z) \neq 0, \quad |z| = 1$$

There exists some  $p_0 < \infty$  such that

$$A_p(z) \neq 0, \quad |z| = 1, \quad p > p_0.$$

- c) Correlation match: By choosing  $p > p_0$ , the difference between the actual covariances and the covariances realized by the model can be made arbitrarily small. The correlation match is exact either at  $p = \infty$  or at  $p = p_0$  if  $\{u\}$  is a  $p_0$  order AR sequence.

The proof of this and all other theorems are given in the Appendix.

Remarks: Unlike the causal (AR) model, it is not true in general for the non-causal models that stability of the  $p^{\text{th}}$  order model guarantees stability of the  $(p+1)^{\text{st}}$  order model. This only holds true for  $p > p_0$  where  $p_0$  is such that  $\inf[A(\omega)] - \theta_{p_0} > 0$  (see A-14). The behavior of 2-D non-causal models is similar to the 1-D case, and the method of analysis is almost parallel and is presented in the next section.

#### IV. TWO DIMENSIONAL NON-CAUSAL MODELS

A Gaussian random field  $\{u(i,j)\}$  has the non-causal MVR

$$u(i,j) = \bar{u}(i,j) + \varepsilon(i,j) \quad (40)$$

$$\bar{u}(i,j) = \sum_{(m,n) \in \hat{S}_3} a(m,n)u(i-m,j-n) \quad (41)$$

where  $\bar{u}(i,j)$  is the minimum variance prediction estimate of  $u(i,j)$  based on all the past and future, and  $\varepsilon(\{i,j\})$  is the prediction error field. From (17b) and in analogy with (26)-(29) for the one dimensional case, we have

$$r(k,\ell) = \sum_{(m,n) \in \hat{S}_3} a(m,n)r(k-m,\ell-n) + \beta^2 \delta(k,\ell), \quad \forall(k,\ell) \quad (42)$$

$$S(z_1, z_2) = \beta^2 / A(z_1, z_2); \quad A(z_1, z_2) = 1 - \sum_{(m,n) \in \hat{S}_3} a(m,n)z_1^{-m}z_2^{-n} \quad (43)$$

$$a(m,n) = a(-m,-n) \quad (44)$$

The covariance of  $\{\epsilon(k,\ell)\}$  are given by

$$r_{\epsilon}(k,\ell) = \beta^2 \delta(k,\ell) - \beta^2 \sum_{(m,n) \in \hat{S}_3} a(m,n) \delta(k-m, \ell-n)$$

or

$$S_{\epsilon}(z_1, z_2) = \beta^2 A(z_1, z_2) \quad (45)$$

implying that  $A(z_1, z_2)$  is not a whitening filter for  $\{u(k,\ell)\}$ .

The variance of the prediction error is given by

$$\beta^2 = 1/\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S^{-1}(\omega_1, \omega_2) d\omega_1 d\omega_2 \quad (46)$$

Paralleling the discussion leading to the definition of 1-D non-causal models, we have a similar definition for the 2-D case.

Definition: A  $(p,q)^{th}$  order minimum variance non-causal model for  $\{u(k,\ell)\}$  using the prediction window  $\hat{W}_3$  is a finite order MVR for another process  $\{\tilde{u}(k,\ell)\}$ .

Realization: With

$$u(i,j) = \sum_{(m,n) \in \hat{W}_3} a_{p,q}(m,n) u(i-m, j-n) + \epsilon_{p,q}(i,j) \quad (47)$$

the  $\{a_{p,q}(m,n)\}$  are chosen to minimize  $E[\epsilon_{p,q}^2(k,\ell)] = \beta_{p,q}^2$ . The associated orthogonality condition yields the realization equation

$$r(k,\ell) = \sum_{(m,n) \in \hat{W}_3} a_{p,q}(m,n) r(k-m, \ell-n) + \beta_{p,q}^2 \delta(k,\ell), \quad (k,\ell) \in W_3 \quad (48)$$

In matrix notation, equation (48) can be written as

$$R_{p,q} a_{p,q} = \beta_{p,q}^2 1_{p,q} \quad (49a)$$

where  $R_{p,q}$  is a doubly block Toeplitz, symmetric matrix defined as



$$S_{\hat{\epsilon}}(z_1, z_2) = \beta_{p,q}^2 A_{p,q}(z_1, z_2), \text{ where}$$

$$A_{p,q}(z_1, z_2) \triangleq 1 - \sum_{(m,n) \in \hat{W}_3} a_{p,q}(m,n) z_1^{-m} z_2^{-n}$$

is chosen to drive the finite order MVR

$$\tilde{u}(i,j) = \sum_{(m,n) \in \hat{W}_3} a_{p,q}(m,n) \tilde{u}(i-m, j-n) + \tilde{\epsilon}(i,j)$$

which is the model for  $\{u(i,j)\}$

Some properties of 2-D non-causal models are similar to those of 1-D models, viz;

$$\left. \begin{aligned} 1) \quad a_{p,q}(m,n) &= a_{p,q}(-m,-n) \\ a_{p,q}(-m,n) &= a_{p,q}(m,-n) \end{aligned} \right\} \quad (m,n) \in \hat{W}_3$$

$$2) \quad \beta_{p,q}^2 \geq \beta_{p+1,q}^2, \quad \beta_{p,q+1}^2 \geq \beta_{p+1,q+1}^2$$

3) If  $\{u(k,l)\}$  is a Markov field with SDF given by [15]

$$S(z_1, z_2) = c / \sum_{(m,n) \in W_3} b(m,n) z_1^{-m} z_2^{-n}$$

then the  $(p,q)$  order non-causal model realizes this SDF exactly. In other words, it is the non-causal MVR of the field, satisfying the orthogonality condition over  $S_3$ . In general, the  $(p,q)$  order non-causal model can be expected to realize an arbitrary SDF only approximately. This is due to the fact that orthogonality is satisfied only over the window  $W_3$  which is a subset of  $S_3$ . Increasing the order of the models may therefore be expected to give better correlation match and spectral fit.

The following theorem shows finite order, stable MVRs can be realized to achieve arbitrarily close correlation or spectral match.

Theorem 2: Let  $\{u(k, \ell)\}$  be a Gaussian random field with PA SDF  $S(z_1, z_2)$ .

Then the following may be proven:

a) Convergence of PEFs: The sequence of non-causal PEFs

$$A_{p,q}(z_1, z_2) = 1 - \sum_{m,n \in \hat{W}_3} a_{p,q}(m,n) z_1^{-m} z_2^{-n}$$

converges uniformly on  $|z_1| = 1, |z_2| = 1$  to the analytic limit PEF  $A(z_1, z_2)$ , i.e.

$$\lim_{(p,q) \rightarrow \infty} A_{p,q}(z_1, z_2) = A(z_1, z_2) \quad , \quad |z_1| = |z_2| = 1$$

The sequence of prediction error variances  $\{\beta_{p,q}^2\}$  converges monotonically to a positive lower limit,  $\beta^2$ .

$$\lim_{(p,q) \rightarrow \infty} \beta_{p,q}^2 = \beta^2 = \left[ \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S^{-1}(\omega_1, \omega_2) d\omega_1 d\omega_2 \right]^{-1}$$

b) Stability: A  $(p,q)$  order non-causal model is stable if and only if

$$A_{p,q}(z_1, z_2) = 0 \quad , \quad |z_1| = |z_2| = 1$$

There exists some  $(p_0, q_0) < \infty$  such that

$$A_{p,q}(z_1, z_2) \neq 0 \quad , \quad |z_1| = |z_2| = 1, (p,q) > (p_0, q_0)$$

c) Correlation match: By choosing  $(p,q) > (p_0, q_0)$ , the difference between the actual correlations and the correlations realized by the model can be made arbitrarily small. At  $(p,q) = \infty$ , the correlation match is exact.

Remarks: The above theorem is similar to the theorem for 1-D non-causal models except for one difference. For the 2-D case, we have been able to prove uniform convergence of the PEFs to the limit PEF only on the unit

circles, whereas for the 1-D case, uniform convergence is obtained in a neighborhood of the unit circle. This is due to the complexity introduced by the added dimension. In any case, the result is still general since for stability and correlation matching considerations, only convergence on the unit circle is required.

## V. SEMI-CAUSAL MODELS

Semi-causal models are recursive in one of the dimensions and non-recursive in the other. In the following, we investigate methods of realizing finite order semi-causal models that will match a given SDF arbitrarily closely. The realization equations may be found in [10], but we present them here for completeness. Unlike non-causal models which do not require spectral factorization for realization, semi-causal models are realized by factorization of  $S(z_1, z_2)$  in one of the variables, corresponding to the causal dimension. The two methods of factorization that we consider are distinct from one another. In the first method, the SDF on its entire region of support is used to realize the model. We present some theorems and use a Levinson type computational algorithm [8] to obtain rational models.

In the second method, covariances given on finite windows are used to design finite order models by solving a set of linear block Toeplitz equations. This is similar to the 2-D non-causal case and similar convergence results for the models are proven.

### Characterization of Semi-causal MVRs:

The semi-causal MVR for a Gaussian random field  $\{u(i, j)\}$  can be written as

$$u(i, j) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} a(m, 0)u(i-m, j) + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} a(m, n)u(i-m, j-n) + \epsilon(i, j) \quad (51)$$

The orthogonality condition (17b) becomes

$$r(k, \ell) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} a(m, 0) r(k-m, \ell) + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} a(m, n) r(k-m, \ell-n) + \beta^2 \delta(k, \ell), \quad k \geq 0, \forall \ell \quad (52)$$

Defining

$$a_0(z_1) \triangleq 1 - \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} a(m, 0) z_1^{-m} \quad (53a)$$

It can be shown that  $a(m, 0)$  form a positive definite sequence, and that  $a(m, 0) = a(-m, 0)$ . This implies that  $a_0^{-1}(z_1)$  exists. Hence we may define

$$a_n(z_1) \triangleq \sum_{m=-\infty}^{\infty} a(m, n) z_1^{-m}, \quad \hat{a}_n(z_1) \triangleq a_n(z_1) / a_0(z_1), \quad n \geq 1 \quad (53b)$$

and taking z-transform on both sides of (52) w.r.t.  $k$ , we obtain

$$r_\ell(z_1) = \sum_{n=1}^{\infty} \hat{a}_n(z_1) r_{\ell-n}(z_1) + \beta^2 a_0^{-1}(z_1) \delta(\ell), \quad \ell \geq 0 \quad (54)$$

In matrix form, the above can be written as

$$\begin{bmatrix} r_0(z_1) & r_1(z_1^{-1}) & r_2(z_1^{-1}) & \dots & \dots & \dots \\ r_1(z_1) & & & & & \\ r_2(z_1) & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \end{bmatrix} \begin{bmatrix} 1 \\ -a_1(z_1) \\ -a_2(z_1) \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} b(z_1) \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}; \quad b(z_1) \triangleq \beta^2 a_0^{-1}(z_1) \quad (55)$$

$\begin{matrix} \nearrow & \nearrow & \nearrow & & & \\ & \nearrow & \nearrow & \nearrow & & \\ & & \nearrow & \nearrow & \nearrow & \\ & & & \nearrow & \nearrow & \\ & & & & \nearrow & \\ & & & & & \nearrow \end{matrix}$

In the following we prove that an SDF may be uniquely factored as

$$S(z_1, z_2) = \frac{\beta^2 a_0(z_1)}{A(z_1, z_2) A(z_1^{-1}, z_2^{-1})} \quad (56a)$$

where

$$A(z_1, z_2) = 1 - \sum_{(m,n) \in \hat{S}_2} a(m,n) z_1^{-m} z_2^{-n}, \quad a_0(z_1) = A(z_1, \infty) \quad (56b)$$

Then we show a computational procedure for solving (55) recursively to obtain the factorization.

Theorem 3: Let the SDF  $S(z_1, z_2)$  be PA. Then it may be uniquely factored as in (56), where the PEF  $A(z_1, z_2)$  is semi-minimum phase.

The following theorem gives a computational procedure similar to the 1-D Levinson algorithm which recursively solves (54) to obtain the semi-causal factors guaranteed in Theorem 3.

Theorem 4: Given that  $S(z_1, z_2)$  is PA, the system of equations (55) can be solved recursively according to the procedure given below for  $|z_1| = 1$ , and  $q \rightarrow \infty$ .

$$\left. \begin{array}{ll} \hat{a}_{0,0} = 1 & (a) \\ \rho_1(z_1) = r_1(z_1)/r_0(z_1) & (b) \\ b_0(z_1) = r_0(z_1) & (c) \\ \text{For } n = 1, 2, \dots & \\ \hat{a}_{n,0} = 1 & (d) \\ \hat{a}_{n,i}(z_1) = \hat{a}_{n-1,i}(z_1) - \rho_n(z_1) \hat{a}_{n-1,n-i}(z_1^{-1}); & (e) \\ \quad \quad \quad i=1, 2, \dots, n-1 & \\ \hat{a}_{n,n}(z_1) = \rho_n(z_1) & (f) \\ b_n(z_1) = b_{n-1}(z_1) [1 - \rho_n(z_1) \rho_n(z_1^{-1})] & (g) \\ \rho_n(z_1) = [r_n(z_1) - \sum_{i=1}^{n-1} \hat{a}_{n-1,i}(z_1) r_{n-i}(z_1)] / b_{n-1}(z_1) & (h) \end{array} \right\} \quad (57)$$



At each stage of the recursion,

$$\hat{A}_n(z_1, z_2) = 1 - \sum_{\ell=1}^n \hat{a}_{n,\ell}(z_1) z_2^{-\ell}$$

is analytic minimum phase and

$$A_n(z_1, z_2) = a_{n,0}(z_1) \hat{A}_n(z_1, z_2)$$

is analytic semi-minimum phase. Here,

$$a_{n,0}(z_1) \triangleq \beta_n^2 / b_n(z_1) \triangleq 1 - \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} a_{n,0(m)} z_1^{-m}$$

At the qth step of the above recursions, the matrix equation to be solved is

$$\begin{bmatrix} r_0(z_1) & r_1(z_1^{-1}) & \dots & r_q(z_1^{-1}) \\ r_1(z_1) & & & \\ r_2(z_1) & & & \\ \vdots & & & \\ r_q(z_1) & & & \end{bmatrix} \begin{bmatrix} 1 \\ -\hat{a}_{q,1}(z_1) \\ -\hat{a}_{q,2}(z_1) \\ \vdots \\ \hat{a}_{q,q}(z_1) \end{bmatrix} = \begin{bmatrix} b_q(z_1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |z_1|=1, \quad (58)$$

Theorems 3 and 4 provide the theoretical basis for designing semi-causal MVRs given the SDF of the random field. The theorems also point the way to obtaining stable, finite order models from the generally irrational MVRs guaranteed by the theorems. Rationalization is achieved in two steps: First, given the  $\{r_\ell(z_1)\}$ , the recursions in (57) are run to a value of  $q$  such that the PEFs obtained approximate the SDF to a desired degree of accuracy. The behaviour of the sequence  $\{\beta_q^2\}$  can be used as an indicator of the goodness of spectral fit. The recursions can be stopped when the decrease in  $\beta_q^2$  becomes marginal, thereby determining the finite order in the  $z_2$  variable. However, the PEFs are still irrational in the  $z_1$  variable.

As far as the second stage of approximation is concerned, an attempt to rationalize the coefficients  $\{a_n(z_1)\}$  will in general lead to unstable models. This can be circumvented by examining the recursions of (57), which shows that as long as the sequence of analytic functions  $\{\rho_n(z_1)\}_1^q$  have magnitude less than unity on  $|z_1| = 1$ , the filters  $\hat{A}_n(z_1, z_2)$  are guaranteed to be AMP. It is possible to find polynomials  $\{\rho_n^r(z_1)\}$  with magnitude less than unity on  $|z_1| = 1$  which uniformly approximate the analytic functions  $\{\rho_n(z_1)\}$ . For example, a truncation and windowing (by  $W_n(k)$ ) to a suitable length of the Fourier coefficients,  $\{\rho_n(k)\}$  of  $\{\rho_n(\omega_1)\}$  exhibit such behaviour. We adopt this procedure for rationalization. Also, in [8], it is shown that if  $S(z_1, z_2)$  is PA, then the reflection coefficient sequence is exponentially bounded, i.e.

$$|\rho_n(k)| < A \alpha_1^n \alpha_2^{|k|}, \quad n \geq 0, \forall k, \quad 0 < \alpha_1, \alpha_2 < 1$$

This property allows us to truncate  $\rho_n(k)$  to shorter and shorter lengths as  $n$  increases, while retaining stability. The following portions of (57) allow us to reconstruct the finite order model:

$$\left. \begin{aligned} \hat{a}_{n,0}^r(z_1) &= 1, \quad n=0,1,2,\dots,q & (a) \\ \hat{a}_{n,i}^r(z_1) &= \hat{a}_{n-1,i}^r(z_1) - \rho_n^r(z_1) \hat{a}_{n-1,n-i}^r(z_1^{-1}), & (b) \\ & \quad i=1,\dots,n-1 \\ \hat{a}_{n,n}^r(z_1) &= \rho_n^r(z_1) & (c) \\ \tilde{b}_0(z_1) &= r_0(z_1) & (d) \\ \tilde{b}_n(z_1) &= \tilde{b}_{n-1}(z_1) [1 - \rho_n^r(z_1) \rho_n^r(z_1^{-1})] & (e) \\ \text{where } \rho_n^r(z_1) &= \sum_{k=-p_n}^{p_n} \rho_n(k) z_1^{-k} & (f) \end{aligned} \right\} \quad (59)$$

Ultimately, however, we are interested in the actual model coefficients,

$a_{q,i}^r = a_{q,0}^r(z_1) \hat{a}_{q,i}^r(z_1)$ . It now remains to find a rational approximation to  $\tilde{b}_q(z_1)$  which will yield  $a_{q,0}^r(z_1)$ . Since  $\tilde{b}_q(z_1)$  is a PA function, we can fit a finite order 1-D non-causal model (see Section III) that will approximate  $\tilde{b}_q(z_1)$  to the desired degree of accuracy, i.e.

$$\tilde{b}_q(z_1) \simeq \tilde{b}_q^r(z_1) = \beta_q^2 / a_{q,0}^r(z_1)$$

This gives the desired finite order model that is semi-minimum phase. Finally, it must be mentioned that the region of support of the predictor coefficients will almost never be rectangular due to the convolutions involved in the recursions [see Eqns. (57)]. Of course, this does not pose any limitation.

As an example, the predictor geometry is shown in Fig. 2 for  $q=3$ , with the sequences  $\{\rho_1(k)\}$ ,  $\{\rho_2(k)\}$ , and  $\{\rho_3(k)\}$  truncated to  $p_1=3$ ,  $p_2=2$ ,  $p_3=1$  [see (59f)], respectively, and for a non-causal model order of  $p=3$ .

A flow chart of the rationalization procedure is shown in Fig. 3 for convenience.

The above method of obtaining stable, semi-causal MVRs assumed complete knowledge of the 2-D covariance sequence in one of the dimensions. The question we address next is the problem of realizing stable semi-causal MVRs given correlations on a finite 2-D window. Except for the difference in prediction geometry, the methods involved are similar to the 2-D non-causal case.

For a prediction window such as  $\hat{W}_2$ , the semi-causal minimum variance model can be written as

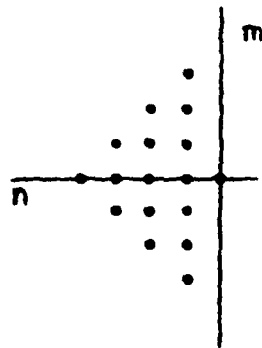
$$u(i,j) = \bar{u}(i,j) + \varepsilon_{p,q}(i,j) \quad (60)$$

$$\bar{u}(i,j) \triangleq \sum_{(m,n) \in \hat{W}_2} a_{p,q}(m,n) u(i-m, j-n) \quad (61)$$

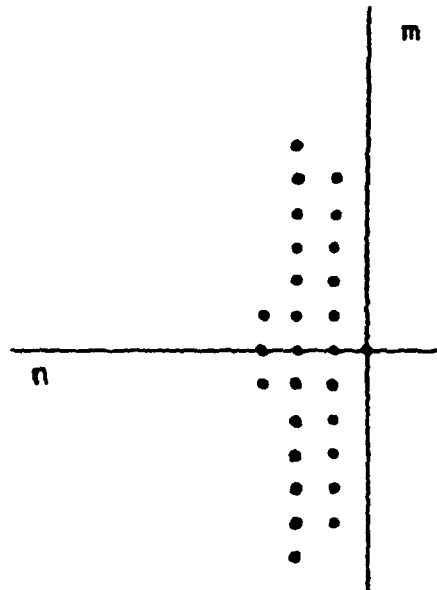
with  $\min[E(\varepsilon_{p,q}^2(i,j))] = \beta_{p,q}^2$ . The normal equations (17c) written for the region

$W_2$  give

$$r(k,l) = \sum_{\substack{m=-p \\ m \neq 0}}^p a_{p,q}(m,0) r(k-m, l) + \sum_{m=-p}^p \sum_{n=1}^q a_{p,q}(m,n) r(k-m, l-n) + \beta_{p,q}^2 \delta(k,l); (k,l) \in W_2 \quad (62)$$

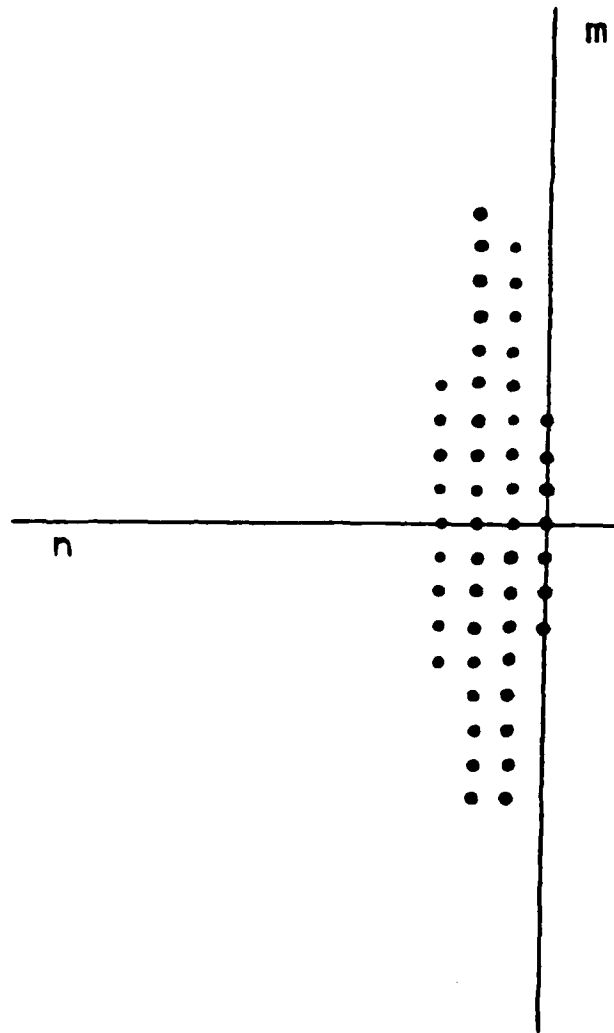


a) Reflection Coefficient Geometry



b) Geometry for  $\{\hat{\lambda}_n(m)\}$

Fig. 2: Example of PEF Support for Semicausal Models  
Via the Rationalization Procedure



c) Prediction Geometry for the Semicausal Model

Fig. 2: Continued

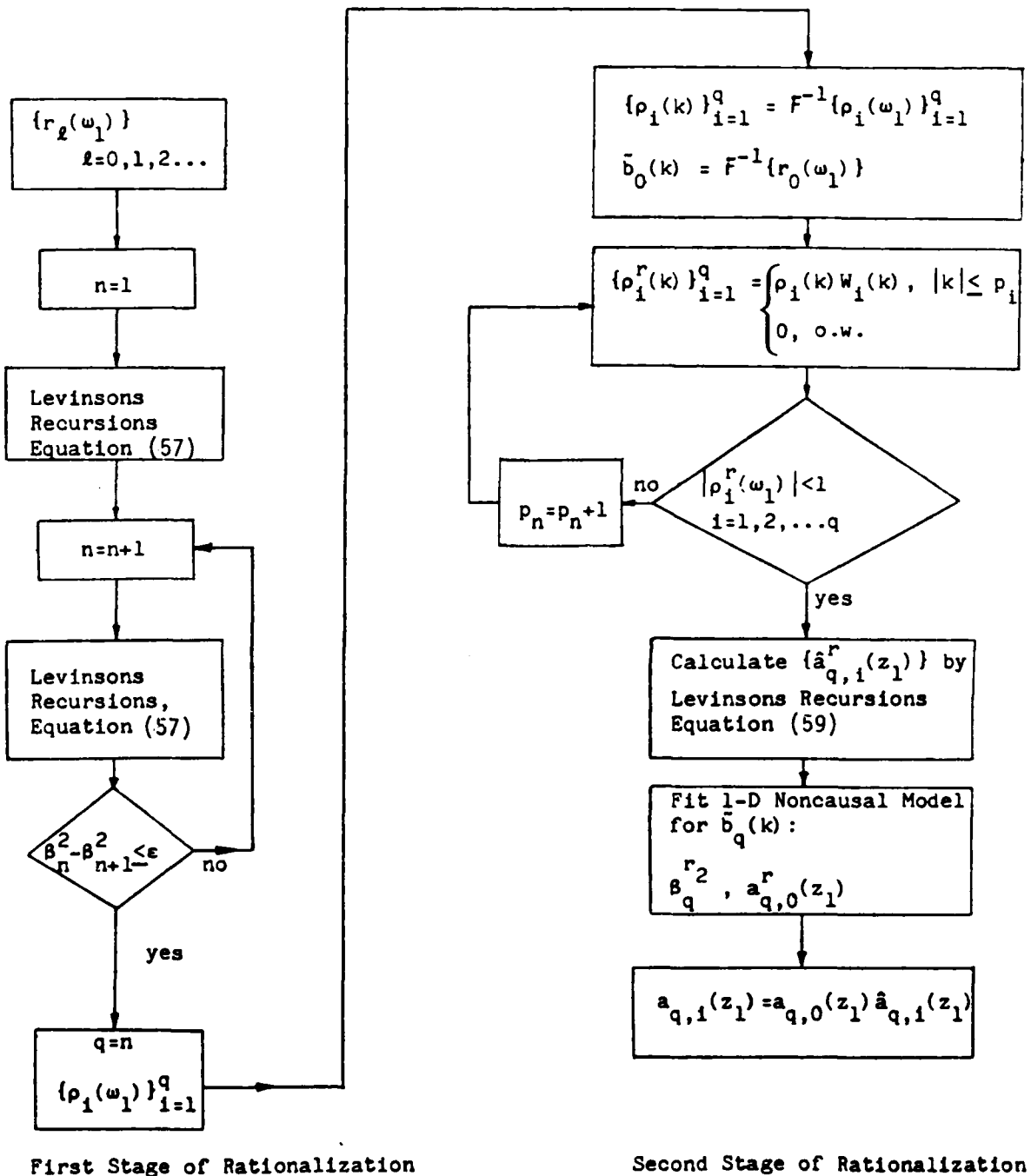


Fig. 3: A Method of Obtaining Finite Order Semicausal Models Via Levinsons Recursions

Defining  $a_{p,q}(0,0) \triangleq 1$ , we may write the above equation in matrix form as

$$R_{p,q} a_{p,q} = \beta_{p,q}^2 \underline{1}_{p,q} \quad (62a)$$

where

$$R_{p,q} \triangleq \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \dots & R_{-q} \\ & R_1 & & & \\ & R_2 & & & \\ & \vdots & & & \\ & R_q & & & \end{bmatrix} \quad (62b)$$

$$R_k \triangleq \begin{bmatrix} r(0,k) & r(-1,k) & \dots & r(-2p,k) \\ & r(1,k) & & \\ & \vdots & & \\ & r(2p,k) & & \end{bmatrix} = R_{-k}^T \quad (62c)$$

$$\underline{a}_{p,q}^T = [\underline{a}_0^T \ \underline{a}_1^T \ \dots \ \underline{a}_q^T] ; \underline{a}_k = [-a(-p,k) \ -a(-p+1,k) \ \dots \ -a(0,k) \ \dots \ -a(p,k)]^T \quad (62d)$$

$$\underline{1}_{p,q}^T = [\underline{1}_p^T \ \underline{0}^T \ \underline{0}^T \ \dots \ \underline{0}^T] ; \underline{1}_p = [0 \ 0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0]^T \quad (62e)$$

(p+1<sup>st</sup> entry)

and  $\beta_{p,q}^2 > 0$  exists for (62a) given by

$$\beta_{p,q}^2 = 1/[R_{p,q}^{-1}]_{p+1,p+1} \quad (63a)$$

$$a_{p,q}(m,n) = \beta_{p,q}^2 [R_{p,q}^{-1}]_{k,p+1} \quad ; \quad k = n(2p+1)+m+p+1 \quad (63b)$$

From equations (62), it is evident that covariances from a window twice the size of  $W_2$  are needed to calculate  $\{a_{p,q}(m,n)\}$  on  $\hat{W}_2$ . Hence, there is no unique correspondence between the predictor coefficients and the covariances realized by the model. Also, stability is not guaranteed. These statements were true for non-causal models realized by this method, and as we shall see, will be true for causal models also. Theorems similar to Theorem 2 may be proven for these models also. The method of proof is similar, with minor differences arising from different prediction geometries.

Theorem 5: Let the Gaussian random field  $\{u(k,l)\}$  have a PA SDF,  $S(z_1, z_2)$ .

Then, the following may be proven:

a) Convergence of PEFs: The sequence of semi-causal PEFs

$$A_{p,q}(z_1, z_2) = 1 - \sum_{(m,n) \in \hat{W}_2} a_{p,q}(m,n) z_1^{-m} z_2^{-n}$$

converges uniformly in the region  $Z_1 = \{|z_1| = 1, |z_2| \geq 1\}$  to the analytic semi-minimum phase PEF  $A(z_1, z_2)$ .

The sequence of prediction error variances  $\{\beta_{p,q}^2\}$  converges monotonically to a positive lower limit,  $\beta^2$

$$\lim_{(p,q) \rightarrow \infty} \beta_{p,q}^2 = \beta^2 = 1 / \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} d^{-1}(\omega_1) d\omega_1 \right], \quad d(\omega_1) = \beta^2 a_0^{-1}(\omega_1)$$

b) Stability: A  $(p,q)$  order semi-causal model is stable iff

$$A_{p,q}(z_1, z_2) \neq 0, \quad (z_1, z_2) \in Z_1$$

There exists some  $(p_0, q_0) < \infty$  such that

$$A_{p,q}(z_1, z_2) \neq 0, \quad (z_1, z_2) \in Z_1, \quad (p,q) > (p_0, q_0)$$



- c) Correlation match: By choosing  $(p,q) > (p_0,q_0)$ , the difference between the actual correlations and the correlations realized by the model can be made arbitrarily small. At  $(p,q) = \infty$ , the correlation match is exact.

## VI. 2-D CAUSAL MODELS

Causal models are recursive in both dimensions with an ordered definition of past and future. Realization of these models from the SDF  $S(z_1, z_2)$  requires factorization in both,  $z_1$  and  $z_2$  variables. Similar to the semi-causal case, we consider two methods of obtaining these models. Realization of these models from the SDF on its entire region of support using the LP approach has received attention in the literature [7-8]. Obtaining rational models has been considered in [8] by way of finite support reflection coefficient representation. The method presented here rationalizes the reflection coefficients obtained in the 2-D Levinson recursions to obtain finite order models. The other method of obtaining rational models is to use covariances given on a finite window, and solve a finite set of linear equations. Convergence and stability results for models realized by this method are similar to the non-causal and semi-causal results.

### Characterization of Causal MVRs:

The causal MVR for a Gaussian random field  $\{u(i,j)\}$  is

$$u(i,j) = \sum_{m=1}^{\infty} a(m,0)u(i-m,j) + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} a(m,n)u(i-m,j-n) + \epsilon(i,j) \quad (64)$$

The corresponding normal equations

$$r(k,\ell) = \sum_{m=1}^{\infty} a(m,0)r(k-m,\ell) + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} a(m,n)r(k-m,\ell-n) + \beta^2 \delta(k,\ell), \quad (k,\ell) \in S_1 \quad (65)$$

reduce in the  $z_1$ -transform domain to

$$r_\ell(z_1) = \sum_{n=1}^{\infty} \hat{a}_n(z_1) r_{\ell-n}(z_1) + \beta^2 a_0(z_1) a_0^{-1}(z_1^{-1}) \delta(\ell), \quad \ell \geq 0 \quad (66)$$

where  $\hat{a}_n(z_1)$ ,  $a_n(z_1)$ ;  $n=1,2,\dots$  are defined in (53b) and

$$a_0(z_1) \triangleq 1 - \sum_{m=1}^{\infty} a(m,0) z_1^{-m} \quad (67)$$

The proof of the existence of a unique solution to (66), which is analytic minimum phase is similar to Theorem 3.

The procedure given in Theorem 4 can be directly used to find the causal MVR except for the following differences:

- 1)  $\{a_{n,0}(z_1)\}$  are minimum phase functions obtained by a second stage of factorization in the  $z_1$  variable of the positive analytic functions  $\{b_n(z_1)\}$  such that

$$b_n(z_1) = \beta_n^2 / a_{n,0}(z_1) a_{n,0}(z_1^{-1}) \quad (68)$$

- 2) The PEFs  $\{A_n(z_1, z_2)\}$  defined as

$$A_n(z_1, z_2) = a_{n,0}(z_1) \hat{A}_n(z_1, z_2) \quad (69)$$

are now analytic minimum phase.

- 3) The rationalization procedure is the same as for semi-causal models, except that a 1-D causal model is used to fit  $\tilde{b}_q(z_1)$  as opposed to a non-causal model.

Now we consider the problem of finding finite order causal models, given covariances on a finite window. The normal equations for a  $(p,q)$  order causal model is

$$r(k, \ell) = \sum_{m=1}^p a_{p,q}(m,0) r(k-m, \ell) + \sum_{m=-p}^p \sum_{n=1}^q a_{p,q}(m,n) r(k-m, \ell-n) = \beta^2 \delta(k, \ell), \quad (k, \ell) \in W_1 \quad (70)$$

In matrix notation, the above equation can be written as

$$R'_{p,q} a_{p,q} = \beta_{p,q}^2 \underline{1}_{p,q} \quad (71a)$$

where

$$R'_{p,q} \triangleq \begin{bmatrix} R'_0 & R'_{-1} & R'_{-2} & \dots & R'_{-q} \\ R'_1 & R'_0 & R'_{-1} & \dots & R'_{-q+1} \\ R'_2 & R'_1 & & & \\ \vdots & \vdots & & & \\ R'_q & R'_{q-1} & & & \end{bmatrix} \quad ,$$

$R'_k$  is defined in (62c) and

$$R'_k = \begin{bmatrix} r(p,k) & r(p-1,k) & \dots & r(-p,k) \\ \vdots & & & \\ r(2p,k) & r(2p-1,k) & \dots & r(0,k) \end{bmatrix} \quad , \quad R'_{-k} = R'^T_k \quad (71b)$$

$$\underline{a}_{p,q}^T \triangleq [\underline{a}_0^T \underline{a}_1^T \underline{a}_2^T \dots \underline{a}_q^T] \quad , \quad \underline{a}_0^T \triangleq [1 \ -a_{p,q}(1,0) \ \dots \ -a_{p,q}(p,0)]^T \quad ,$$

$$a_{p,q}(0,0) \triangleq 1$$

$$\underline{a}_i^T \triangleq [-a_{p,q}(-p,i) \ \dots \ -a_{p,q}(0,i) \ \dots \ -a_{p,q}(p,i)]$$

$$\underline{1}_{p,q}^T \triangleq [\underline{1}_p^T \ \underline{0}^T \ \underline{0}^T \ \dots \ \underline{0}] \quad , \quad \underline{1}_p^T = [1 \ 0 \ \dots \ 0] \quad (71c)$$

Given that  $S(\omega_1, \omega_2) > 0$ , a unique solution to the system of equations (71) exists. Considerations such as convergence of PEFs, stability and correlation match follow similar lines of reasoning as in the non-causal and semi-causal case. We briefly state the results:

- 1) The sequence of PEFs

$$A_{p,q}(z_1, z_2) = 1 - \sum_{m=1}^p a_{p,q}(m,0)z_1^{-m} - \sum_{m=-p}^p \sum_{n=1}^q a_{p,q}(m,n)z_1^{-m}z_2^{-n}$$

converge uniformly to the limit PEF  $A(z_1, z_2)$  in the region

$$Z = \{|z_1| = 1, |z_2| \geq 1\} \cup \{|z_1| \geq 1, z_2 = \infty\}$$

The sequence of prediction error variances converges monotonically to a positive lower limit

$$\lim_{(p,q) \rightarrow \infty} \beta_{p,q}^2 = \beta^2 = \exp\left[\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log S(\omega_1, \omega_2) d\omega_1 d\omega_2\right]$$

- 2) For some  $(p,q) > (p_0, q_0)$ ,  $A_{p,q}(z_1, z_2) \neq 0$  for  $(z_1, z_2) \in Z$ , so that the causal models of order  $(p,q) > (p_0, q_0)$  are stable.
- 3) For  $(p,q) > (p_0, q_0)$  the correlation match can be made arbitrarily small.

## VII. EXPERIMENTAL RESULTS AND COMPARISONS

The following provides a verification of the various theoretical results obtained for 2-D stochastic models in the preceding sections. The results are demonstrated for a covariance model that is often used in image processing viz; the exponential isotropic covariance,  $r(k, \ell) = \rho^{\sqrt{k^2 + \ell^2}}$ . For our simulations, we have used  $\rho=0.9$ . A 5x5 portion of this covariance sequence is shown in Fig. 4. Only the entries in the positive quadrant are given, since the other quadrants can be filled in by symmetry. The SDF corresponding to this covariance sequence is irrational and satisfies the PA assumption. Wherever required, z-transforms were evaluated on the unit circle(s) using 256 point DFTs in 1-D and 256x256 DFTs in 2-D.

Non-causal models were realized for this covariance sequence by solving the linear block Toeplitz equations of (49) using an efficient recursive algorithm [16]. The SDF corresponding to a  $(p,q)$  order model is

.551	.591	.624	.648	.656	
.591	.640	.684	.717	.729	
.624	.684	.742	.790	.810	k
.648	.717	.790	.862	.900	
.656	.729	.810	.900	1.000	
		l			(0,0)

Fig. 4: Covariances Corresponding to  $r(k,l) = .9^{\sqrt{k^2+l^2}}$   
on the Quarter Plane

$$S_{p,q}(z_1, z_2) = \frac{\beta_{p,q}^2}{A_{p,q}(z_1, z_2)}$$

$$\text{where } A_{p,q}(z_1, z_2) = 1 - \sum_{\substack{m=-p \\ (m,n) \neq (0,0)}}^p \sum_{n=-q}^q a_{p,q}(m,n) z_1^{-m} z_2^{-n}.$$

The covariances realized by the model,  $\{r_{p,q}(k, \ell)\}$  are obtained as the inverse transform of  $S_{p,q}(z_1, z_2)$ .

Figure 5 shows the PEFs and the covariance mismatch for (1,1) and (1,2) order non-causal models. Examination of the PEF for the (1,1) model seems to suggest a four point model, but this is belied by the substantial covariance mismatch obtained. The (1,2) model gives a smaller mismatch, and  $\beta_{1,2}^2 < \beta_{1,1}^2$ . Results obtained for progressively larger models confirm the results of Theorem 2.

Infinite order spectral factorization was carried out to obtain the semi-causal MVR using the Wiener-Doob factorization method. Details of this method can be found in [10]. The coefficients of the PEF corresponding to the MVR are shown in Fig. 6. Examination of the coefficients  $\{a(m,0)\}$  indicate that  $\epsilon\{(i,j)\}$  is very nearly a first order moving average process in the non-causal dimension.

Semi-causal models were realized using the rationalization procedure of Section V. Fig. 7 shows the PEF of the rational model obtained by using second order in the causal dimension ( $q=2$ ). The infinite length reflection coefficient sequences  $\{\rho_1(k)\}$  and  $\{\rho_2(k)\}$  were truncated to 5 terms and 3 terms, respectively, i.e.

$$\rho_1^r(k) = \begin{cases} \rho_1(k) & , \quad |k| \leq 2 \\ 0 & , \quad \text{o.w.} \end{cases} \quad \rho_2^r(k) = \begin{cases} \rho_2(k) & , \quad |k| \leq 1 \\ 0 & , \quad \text{o.w.} \end{cases}$$

A second order ( $p=2$ ) 1-D non-causal model was used to fit  $\tilde{b}_2(z_1)$ , giving a total of 23 coefficients for the semi-causal model. For this case, the PEF is given as

$$\begin{array}{cccc} -.0006 & -.2518 & -.0006 & 1 \\ -.2518 & 1.0000 & -.2518 & 0 \\ -.0006 & -.2518 & -.0006 & -1 \\ 1 & 0 & -1 & \end{array} \quad m$$

$$B_{1,1}^2 = .039527$$

a) PEF Coefficients Corresponding to  $(p,q) = (1,1)$  Noncausal Model

.0278	-.0345	-.2417	-.0345	.0278	1	
.0209	-.2700	1.0000	-.2700	.0209	0	m
.0278	-.0345	-.2417	-.0345	.0278	-1	
-2	-1	0	1	2		
		n				

$$B_{1,2}^2 = .038533$$

b) PEF Coefficients Corresponding to  $(p,q) = (1,2)$  Noncausal Model

.413	.420	.427	.433	.435		.324	.180	.053	-.033	-.065	
.420	.431	.444	.455	.459		.337	.190	.061	-.029	-.062	
.427	.444	.465	.484	.491	k	.350	.204	.076	-.014	-.046	k
.433	.455	.484	.516	.526		.362	.218	.094	.008	-.025	
.435	.459	.491	.526	.537		.366	.223	.099	.015	-.016	
	$\ell$			(0,0)			$\ell$			(0,0)	

c)  $\{r(k, \ell) - r_{1,1}(k, \ell)\}$

d)  $\{r(k, \ell) - r_{1,2}(k, \ell)\}$

Fig. 5: PEFs and Covariance Mismatches for  $(p,q) = (1,1)$  and  $(1,2)$  Noncausal Models

	.	.	.	.	
	.	.	.	.	
	.	.	.	.	
	.	.	.0010	-.0016	
	.	.	.0015	-.0024	
	.	.0004	.0026	-.0039	
	.0004	.0052	.0259	.0019	2
	.0010	.0168	-.0385	-.3866	1
. . . . .	.0014	.0282	-.2753	1.0	0 m
	.0010	.0168	-.0385	-.3866	-1
	.0004	.0052	.0259	.0019	-2
	.	.0006	.0056	-.0063	
	.	.0004	.0026	-.0039	
	.	.	.0015	-.0024	
	.	.	.0010	-.0016	
	.	.	.	.	
	.	.	.	.	
	3	2	1	0	

n

$$\beta^2 = .093242$$

Fig. 6: Infinite Order Semicausal PEF Obtained by Wiener-Doob Factorization



	-.0002			
	-.0045			
.0031	-.0359			
.0312	-.0556	-.0453	2	
-.0307	.0177	-.4702	1	
-.0278	.2632	1	0	m
-.0107	.0177	-.4702	-1	
.0312	-.0556	-.0453	-2	
.0031	-.0359			
	-.0045			
	-.0002			
2	1	0		
n				

$$B^2 = .0944089$$

a) Coefficients of Semicausal PEF

.329	.338	.340	.334	.321	
.322	.328	.325	.313	.294	
.316	.316	.306	.287	.262	k
.313	.311	.286	.380	.341	
.311	.309	.293	.263	.234	
l					(0,0)

b) Covariance Mismatch,  $\{r(k,l) - \hat{r}(k,l)\}$

Fig. 7: Rationalized Semicausal Model Corresponding to  $q=2$ ,  $\{\rho_1(k)\}$  Truncated to 5 Terms,  $\{\rho_2(k)\}$  Truncated to 3 Terms and  $p=2$ .

$$A(z_1, z_2) = 1 - \sum_{\substack{m=-2 \\ m \neq 0}}^2 a(m,0)z_1^{-m} - \sum_{m=-5}^5 a(m,1)z_1^{-m}z_2^{-1} - \sum_{m=-3}^3 a(m,2)z_1^{-m}z_2^{-2}$$

The SDF realized by this model is then

$$S(z_1, z_2) = \beta^2 \left( 1 - \sum_{\substack{m=-2 \\ m \neq 0}}^2 a(m,0)z_1^{-m} \right) / A(z_1, z_2) A(z_1^{-1}, z_2^{-1})$$

The coefficients  $\{a(m,0)\}$  can be seen to differ appreciably from the 'true' coefficients in Fig. 6, which explains the substantial covariance mismatch in Fig. 7.

By using  $q=1$ , truncating  $\{\rho_1(k)\}$  to 15 terms and using  $p=2$  yielded the semi-causal PEF in Fig. 8, having a total of 24 coefficients. As can be seen, the covariance match here is much better than in the previous case. Also, the coefficients  $\{a(m,0)\}_{-2}^2$  are closer to the true coefficients. This emphasizes the importance of the leading reflection coefficients which have the larger magnitudes. Since [see (59)].

$$\tilde{b}_q(z_1) = b_0(z_1) \prod_{i=1}^q [1 - \rho_i^r(z_1) \rho_i^r(z_1^{-1})] \approx \beta_q^2 / a_{q,0}^r(z_1)$$

the coefficients  $\{a(m,0)\}$  will be most sensitive to  $\{\rho_1(k)\}$ . However, the price paid is large order models.

Semi-causal models are also obtained by solving the set of linear block Toeplitz equations (62). Fig. 9 shows the semi-causal PEFs obtained for model orders  $(p,q) = (1,1), (2,2), (3,3)$ . Examination of the PEFs for orders  $(1,1)$ ,  $(2,2)$  and  $(3,3)$  shows that the PEF coefficients converge quite rapidly to the true coefficients in Fig. 6. Also,  $\beta_{3,3}^2 < \beta_{2,2}^2 < \beta_{1,1}^2$  and approach the limit,  $\beta^2$  of the infinite PEF. The correlation match also gets better and these are shown in Fig. 10.

```

      :
      :
      :
- .0003
- .0016
- .0089
- .0408      - .012      2
. 0156      - .3969      1
. 250      1.      0      m
. 0156      - .3969      -1
- .0408      - .012      -2
- .0089
- .0016
- .0003
      1      0
      :
      :      n
      :
 $\beta^2 = .09398$ 

```

### a) Semicausal PEF Coefficients

.089	.118	.142	.159	.161
.074	.106	.134	.154	.156
.058	.092	.124	.148	.148
.046	.082	.116	.145	.143
.041	.077	.113	.143	.142

b) Covariance Mismatch,  $\{r(k, \ell) - \hat{r}(k, \ell)\}$

Fig. 8: Semicausal Model Obtained Using  $q=1$ , Truncating  $\{\rho_1(k)\}$  to 15 Terms, and  $p=2$ .

.0135	-.3887	1	
-.2546	1.0	0	m
-.0135	-.3887	-1	

1      0  
n

$$\beta^2 = .094543$$

a) PEF Coefficients of  
(p,q)=(1,1) Semicausal  
Model

.0094	.0268	-.0028	2
.0187	-.0388	-.3875	1
.0304	-.2578	1.0	0 m
.0187	-.0388	-.3875	-1
.0094	.0268	-.2238	-2

2      1      0  
n

$$\beta^2 = .093297$$

b) PEF Coefficients of  
(p,q)=(2,2) Semicausal  
Model

.0017	.0011	.0058	-.0097	3
.0009	.0049	.0263	.0018	2
.0017	.0166	-.0384	-.3869	1
.0021	.0281	-.2753	1.0	0 m
.0017	.0166	-.0384	-.3869	-1
.0009	.0049	.0263	.0018	-2
.0017	.0011	.0058	-.0097	-3

3      2      1      0  
n

$$\beta^2 = .0932627$$

c) PEF Coefficients of  
(p,q)=(3,3) Semicausal  
Model

Fig. 9: Semicausal Model PEFs Obtained by Solving  
Finite Order Equations

-1.492	-1.462	-1.438	-1.429	-1.414	.099	.111	.120	.127	.130
-1.577	-1.544	-1.514	-1.492	-1.485	.097	.109	.117	.123	.125
-1.639	-1.600	-1.564	-1.557	-1.530	.095	.107	.115	.119	.120
-1.674	-1.632	-1.589	-1.553	-1.551	.095	.106	.113	.117	.117
-1.687	-1.642	-1.597	-1.559	-1.557	.095	.106	.113	.115	.116

$\ell$

(0,0)

$\ell$

(0,0)

a)  $\{r(k, \ell) - r_{1,1}(k, \ell)\}$

b)  $\{r(k, \ell) - r_{2,2}(k, \ell)\}$

.084	.090	.094	.098	.099
.082	.088	.091	.094	.094
.080	.085	.088	.090	.090
.080	.084	.086	.088	.088
.079	.083	.085	.087	.087

$\ell$

(0,0)

c)  $\{r(k, \ell) - r_{3,3}(k, \ell)\}$

Fig. 10: Covariance Mismatches of the Semicausal Models in Fig. 9

Next, for causal models, the PEF corresponding to the infinite order MVR is shown in Fig. 11. This has been obtained by Wiener-Doob factorization, [10]. Causal models were realized using the rationalization procedure similar to Section V. Recall that the procedure is the same except that instead of fitting a non-causal model to  $\tilde{b}_q(z_1)$ , as in the semi-causal case, a further factorization of  $\tilde{b}_q(z_1)$  into minimum phase and maximum phase factors is involved in the causal case. Fig. 12 shows the PEF of the model obtained by using  $q=2$ . truncating  $\{\rho_1(k)\}$  and  $\{\rho_2(k)\}$  to 5 and 3 terms, respectively, and fitting a second order ( $p=2$ ) 1-D causal model to  $\tilde{b}_2(z_1)$ .

The PEF has 17 coefficients and realizes the SDF given by

$$S(z_1, z_2) = \beta^2 / A(z_1, z_2) A(z_1^{-1}, z_2^{-1})$$

$$A(z_1, z_2) = 1 - \sum_{m=1}^2 a(m,0) z_1^{-m} - \sum_{m=-3}^5 a(m,1) z_1^{-m} z_2^{-1} - \sum_{m=-1}^3 a(m,2) z_1^{-m} z_2^{-2}.$$

From Fig. 12(f) the covariance mismatch is seen to be quite high, though it is less than the corresponding semi-causal case (Fig. 7(b)). Fig. 13 shows the causal PEF coefficients corresponding to  $q = 1$ ,  $\{\rho_1(k)\}$  truncated to 15 terms and  $p = 5$ . The total number of coefficients is 26. The correlation mismatch is much less than the previous case, and better than the corresponding semi-causal case (Fig. 8(b)). Considerations of the effects of reflection coefficient truncation are similar to the semi-causal case.

The PEFs obtained by solution of the linear equations of (88) corresponding to causal model orders of  $(p,q) = (1,1)$ ,  $(2,2)$ , and  $(3,3)$  are shown in Fig. 14. This again verifies the convergence of the PEFs  $\{A_{p,q}(z_1, z_2)\}$  and prediction errors,  $\{\beta_{p,q}^2\}$  to their limiting values in Fig. 11. Fig. 15



	.0004		
	.0109		
.0080	.0671		
.0300	.1380	-.1299	2
.0105	.0582	-.5827	1
.0434	-.3183	1.	0
.0674	-.2091		-1
	-.0669		-2
	-.0034		-3
2	1	0	
	n		

$$B^2 = .1255927$$

a) Coefficients of Causal PEF

.231	.224	.212	.179	.150	
.220	.206	.183	.148	.111	
.208	.191	.160	.118	.073	$\kappa$
.203	.183	.149	.101	.058	
.200	.180	.145	.095	.053	
		$l$			(0,0)

b) Covariance Mismatch,  $\{r(k, \ell) - \hat{r}(k, \ell)\}$

Fig. 12: Rationalized Causal Model Corresponding to  $q=2$ ,  $\{\rho_1(k)\}$  Truncated to 5 Terms,  $\{\rho_2(k)\}$  Truncated to 3 Terms and  $p=2$



```

      :
      .0333
      .0011
      .0013
      .0050
      .0125   -.0197
      .0165   -.0150
      .0247   -.0207
      .0620   -.0146   -2
      .0100   -.5043   -1
      -.3084   1.       0   m
      -.1756   -1
      -.0424
      -.0129
      -.0052
      .0025
      .0014
      .0012
      .
      i       0
      n
  
```

$$\beta^2 = .117803$$

a) Coefficients of Causal PEF

```

      -.011   .015   .034   .046   .044
      -.027   .002   .027   .043   .040
      -.043   -.011   .019   .040   .038 k
      -.054   -.021   .012   .039   .037
      -.059   -.025   .009   .038   .036
                        l           (0,0)
  
```

b) Covariance Mismatch  $\{r(k,l) - \hat{r}(k,l)\}$

Fig. 13: Rationalized Causal Model Obtained Using  $q=1$ ,  $\{p_1(k)\}$  Truncated to 15 Terms and  $p = 5$

.054	-.4963	1	
-.3113	1.0	0	m
-.233		-1	
1	0		

n

$$B_{1,1}^2 = .118936$$

a) PEF Coefficients of  
(p,q) = (1,1) Causal Model.

-.0149	.0377	-.0124	2
-.0280	-.0229	-.4797	1
-.0516	-.3534	1.0	0 m
-.0509	-.216		-1
-.0289	-.1009		-2
2	1	0	
	n		

$$B_{2,2}^2 = .11499$$

b) PEF Coefficients of  
(p,q) = (2,2) Causal Model.

.0035	.0019	.0095	-.0175	3
.0019	.0063	.0370	-.0035	2
.0031	.0235	-.0295	-.4774	1
.0042	.0458	-.3519	1.0000	0 m
.0042	.0427	-.2149		-1
.0039	.0285	-.0715		-2
.0038	.0125	-.0478		-3
3	2	1	0	

n

$$B_{3,3}^2 = .114595$$

c) PEF Coefficients of  
(p,q) = (3,3) Model.

Fig. 14: Causal Model PEFs Obtained by Solving  
Finite Order Equations

.144	.182	.218	.252	.277		.181	.199	.215	.232	.247	
.149	.187	.225	.259	.284		.181	.197	.211	.224	.236	
.156	.194	.233	.267	.286	k	.180	.195	.206	.215	.222	k
.169	.205	.244	.279	.283		.183	.196	.204	.209	.212	
.186	.219	.253	.280	.280		.187	.198	.206	.208	.208	
$\ell$					(0,0)	$\ell$					(0,0)

a)  $\{r(k, \ell) - r_{1,1}(k, \ell)\}$

b)  $\{r(k, \ell) - r_{2,2}(k, \ell)\}$

.130	.134	.147	.156	.164	
.128	.136	.142	.148	.153	
.126	.133	.137	.141	.144	k
.127	.132	.135	.138	.139	
.128	.133	.135	.136	.137	
$\ell$					(0,0)

c)  $\{r(k, \ell) - r_{3,3}(k, \ell)\}$

Fig. 15: Covariance Mismatches for the Causal Models in Fig. 14

shows the covariance mismatches obtained with the above three causal models. The (1,1) causal model seems to provide a surprisingly good covariance match compared to the corresponding semi-causal model (Fig. 10(a)).

#### VIII. CONCLUSIONS

We have proved a new theorem on the properties of 1-D non-causal models which establishes convergence of PEFs, and guarantees model stability and improvement in covariance matching properties after a finite model order.

The spectral factorization methods of [7,8] (for causal models) and [10] (for causal and semi-causal models) realize irrational models by the solution of an infinite set of normal equations. Theorems were proven to establish existence and uniqueness of solutions. The SDFs realized by the Levinson like computational algorithm of [7,8] were shown to converge uniformly to the true SDF. A method of rationalizing the reflection coefficients obtained in this algorithm was used to obtain rational models with guaranteed stability.

A method of realizing 2-D non-causal, semi-causal and causal models by solving a set of linear equations [10] was considered. The advantage of this method is that only a finite set of equations need be solved. However, these models were not guaranteed to be stable or provide covariance match. The results proven in this paper have shown the uniform convergence of the PEF sequence to a limit PEF, from which model stability and improved covariance match were shown to follow after some finite model order. These results enhance the practical appeal of this technique.

Examples of finite order model realizations were shown for an isotropic covariance function, which corroborated the theory developed. For this covariance function, it was found that the non-causal model needed much larger model orders than either the causal or semi-causal models, to provide uniformity

in covariance matching results. The performance of the latter two models were quite good and on par with each other.

# APPENDIX

Definitions: The following definitions and properties are useful in proving the theorems presented in this paper.

a) For a vector

$$\underline{x} = [x(1) \ x(2) \ x(3) \ \dots \ x(N)]^T$$

the Frobenius norm is defined as

$$\|\underline{x}\|_F \triangleq \sqrt{\sum_{i=1}^N x^2(i)} \quad (A-1)$$

If A is a M x N matrix, then

$$\|A\|_F \triangleq \sqrt{\sum_{i=1}^M \sum_{j=1}^N a^2(i,j)} \quad (A-2)$$

It can be proven that

$$\|Ax\|_F \leq \|A\|_F \|x\|_F \quad (A-3)$$

b) The  $L_2$  or 2-norm of  $\underline{x}$  is

$$\|\underline{x}\|_2 \triangleq \sqrt{\sum_{i=1}^N x^2(i)} = \|\underline{x}\|_F \quad (A-4)$$

If A is a M x N matrix then,

$$\|A\|_2 \triangleq \sqrt{\lambda_{\max}(A^T A)} \quad (A-5)$$

where  $\lambda(X)$  denotes an eigenvalue of the matrix X.

If A is square symmetric, then

$$\|A\|_2 = \lambda_{\max}(A) \quad , \quad \|A^{-1}\|_2 = 1/\lambda_{\min}(A) \quad (A-6)$$

Also,

$$\|Ax\|_2 \leq \|A\|_2 \|x\|_2 \quad (A-7)$$

and if  $A$  is  $M \times N$  and  $B$  is  $N \times L$ , then

$$\|AB\|_F \leq \|A\|_2 \|B\|_F \quad (A-8)$$

c) For the sequence of symmetric matrices

$$A_r = \{a(i,j)\} \quad , \quad (i,j) = 1,2,\dots,r \text{ for } r = 1,2,\dots,N$$

let  $\lambda_k(A_r)$ ,  $k = 1,2,\dots,r$  denote the  $k^{\text{th}}$  eigenvalue of  $A_r$  where

$$\lambda_1(A_r) \geq \lambda_2(A_r) \geq \dots \geq \lambda_r(A_r)$$

$$\text{Then } \lambda_{k+1}(A_{i+1}) \leq \lambda_k(A_i) \leq \lambda_k(A_{i+1}) \quad (A-9)$$

The significance of this result [12] is that the smallest eigenvalue of a matrix sequence is non-increasing with size, and is useful in establishing upper bounds.

#### Proof of Theorem 1:

Let  $S(z)$  be analytic in  $Z_1 = \{\gamma < |z| < 1/\gamma, \quad 0 < \gamma < 1\}$ . The PA condition on  $S(z)$  yields i)  $\{r(\ell)\}$  forms a positive definite, exponentially bounded sequence, i.e.  $|r(\ell)| < B\gamma^{|\ell|}$ ,  $\forall \ell$ , ii)  $S^{-1}(z)$  is PA in some

$$Z_2 = \{\alpha < |z| < 1/\alpha, \quad 0 < \alpha < 1\}. \text{ Therefore, if } S^{-1}(z) = \sum_{\ell=-\infty}^{\infty} r^{-}(\ell)z^{-\ell} \quad ,$$

$z \in Z_2$  then  $|r^{-}(\ell)| = A\alpha^{|\ell|}$ ,  $\forall \ell$ . From (27), for the infinite non-causal MVR we have

$$S^{-1}(z) = \frac{1}{\beta^2} \left[ 1 - \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} a(\ell)z^{-\ell} \right] \quad (A-10)$$

which means  $r^{-}(0) = \frac{1}{\beta^2}$ , and  $a(\ell) = -\beta^2 r^{-}(\ell)$ ,  $\forall \ell$ . Since  $r^{-}(\ell)$  is

exponentially bounded, this means  $|a(\ell)| < A\beta^2 \alpha^{|\ell|}$ ,  $\forall \ell$  and, the limit PEF

$A(z) = 1 - \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} a(\ell)z^{-\ell}$  is PA in  $Z_2$ . Now, subtracting the realization equation

(32) for the  $p^{\text{th}}$  order model from (26) for  $|n| \leq p$ , using the definitions in

(33), and defining a vector  $\underline{v}_p$  with elements  $v_p(n) = \sum_{|\ell| > p} a(\ell)r(n-\ell)$ ,  $|n| \leq p$ ,

we get

$$R_p[\underline{a}_p - \underline{a}] = (\beta^2 - \beta_p^2)\underline{1}_p + \underline{v}_p$$

where  $\underline{a} \triangleq [-a(-p) - a(-p+1) \dots -a(-1) \quad 1 \quad -a(1) \dots a(p)]^T$ .

Now, using (33a) and defining

$$\underline{a}'_p \triangleq \frac{\beta^2}{\beta_p^2} \underline{a}_p \quad (\text{A-11})$$

we get from the above equation that

$$\underline{a}'_p - \underline{a} = R_p^{-1} \underline{v}_p$$

Taking  $\|\cdot\|_2$  norms on both sides of the above equation and using (A-7) we get

$$\|\underline{a}'_p - \underline{a}\|_2 \leq \|R_p^{-1}\|_2 \|\underline{v}_p\|_2.$$

Using the bounds for  $\{r(\ell)\}$  and  $\{a(\ell)\}$  and evaluating the norm of  $\underline{v}_p$ , we obtain

$$\|\underline{a}'_p - \underline{a}\|_2 \leq \|R_p^{-1}\|_2 \left[ \frac{2AB\beta^2\alpha\gamma}{1-\alpha\gamma} \sqrt{\frac{1-\gamma^{4p+2}}{1-\gamma^2}} \right] \alpha^p$$

Denoting the term in square brackets by  $C$  and noting that  $|x(i)| \leq \|\underline{x}\|_2$  for

any arbitrary  $\underline{x}$ , the above inequality yields  $|a'_p(k) - a(k)| < C \|R_p^{-1}\|_2 \alpha^p$  for  $|k| \leq p$ . Using

(A-8) and the fact that  $\lambda_{\min}(R_p) \geq \lambda_{\min}(R_{p+1}) \geq \dots \geq R_{\infty} = \inf[S(\omega)] = C'$ ,

[see (A-9)] we obtain  $|a'_p(k) - a(k)| < C'' \alpha^p$ ,  $C'' \triangleq C/C'$ . Since  $\alpha^p \leq \alpha_1^p \alpha_1^{|k|}$

for  $|k| \leq p$  and  $\alpha_1 = \sqrt{\alpha}$ , we finally obtain

$$|a'_p(k) - a(k)| < C'' \alpha_1^p \alpha_1^{|k|}, \quad |k| \leq p \quad (\text{A-12})$$



With the above exponential bound in (A-12), it becomes a simple matter to establish uniform convergence of the sequence of PEFs  $\{A_p(z)\}$  to  $A(z)$  [16].

To see this, we write the expression for  $|A'_p(z) - A(z)|$ , use (A-12) and the bound for  $|a(k)|$  to yield

$$|A'_p(z) - A(z)| < C'' \alpha_1^p \sum_{k=-p}^p \alpha_1^{|k|} |z|^{-k} + A\beta^2 \left[ \sum_{k=p+1}^{\infty} (\alpha|z|)^k + \sum_{k=p+1}^{\infty} (\alpha|z|^{-1})^k \right]$$

In the neighborhood  $Z_3 = \{\frac{\alpha_1}{\delta} < |z| < \frac{\delta}{\alpha_1}, \alpha_1 < \delta < 1\}$  the above bound can be proven to be

$$|A'_p(z) - A(z)| < \epsilon \epsilon^p \quad (A-13)$$

where  $C > 0$  and  $0 < \epsilon < 1$  are constants depending on  $\alpha_1$ ,  $\delta$ , etc. Substituting for  $\underline{a}'_p$  in terms of  $\underline{a}_p$  [see (A-11)], it is easy to prove that

$$|A_p(z) - A(z)| < \frac{C\beta_p^2 \epsilon^p}{\beta^2} + \frac{\beta_p^2 - \beta^2}{C'} \triangleq \theta_p \quad (A-14)$$

where  $C' = \inf[|S(z)|]$  in some neighborhood of  $|z| = 1$ .

Hence,  $\{A_p(z)\}$  converge uniformly to  $A(z)$  in some neighborhood of  $|z| = 1$ . It has been proven that  $\{\beta_p^2\}$  is a monotone non-increasing sequence. Since  $S(z)$  is PA,  $\beta^2$  in (30) exists and the sequence converges to this limit.

**Stability:** Since  $S(\omega) > 0$  and analytic,  $A(\omega)$  is analytic and has no zeros on the unit circle. From (A-14) we obtain  $A_p(\omega) > A(\omega) - \theta_p$ . Hence, a model order  $p_0$  can be chosen so that for all  $p \geq p_0$ ,  $\inf[A(\omega)] - \theta_p > 0$ , guaranteeing stability of all non-causal models of order greater than  $p_0$ .

#### Correlation Match:

$$\text{For } p > p_0, |S(\omega) - S_p(\omega)| = \left| \frac{\beta^2}{A(\omega)} - \frac{\beta_p^2}{A_p(\omega)} \right| < \frac{C\epsilon^p}{\beta^2} |S(\omega)S_p(\omega)| < K'\epsilon^p$$

In writing in the above we have used (A-11), (A-12), the least upper bound for  $S(\omega)$  and the fact that  $S_p(\omega)$  can be uniformly bounded from above for  $p > p_0$ . If

$\{r_p(k)\}$  denote the covariances realized by the  $p^{\text{th}}$  order model, then

$$|r(k) - \hat{r}_p(k)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |S(\omega) - S_p(\omega)| d\omega < K' \epsilon^p.$$

Thus, the correlation match gets better (exponentially so) with increasing model order.

Proof of Theorem 2:

Subtracting the realization equation (42), for the infinite order MVR from (48), the realization equation for the  $(p,q)$  order model, for  $(k,\ell) \in W_3$  and using the definitions in (49), we obtain

$$R_{p,q}[\underline{a}_{p,q} - \underline{a}] = [\beta^2 - \beta_{p,q}^2] \underline{1}_{p,q} + \sum_{m=-p}^p \sum_{|n|>q} a(m,n) r(k-m, \ell-n) + \sum_{n=-q}^q \sum_{|m|>p} a(m,n) r(k-m, \ell-n) \quad (k,\ell) \in W_3$$

where  $\underline{a}$  is the vector of elements  $\{a(m,n)\}$ ,  $(m,n) \in W_3$  arranged corresponding to the elements of  $\underline{a}_{p,q}$ . The summations are interpreted as vectors with elements indexed in  $(k,\ell)$ . Defining

$$\underline{a}'_{p,q} \triangleq \beta^2 / \beta_{p,q}^2 \underline{a}_{p,q} \quad (A-15)$$

we can write

$$\underline{a}'_{p,q} - \underline{a} = R_{p,q}^{-1} \left[ \sum_{m=-p}^p \sum_{|n|>q} (\cdot) + \sum_{n=-q}^q \sum_{|m|>p} (\cdot) \right], \quad (k,\ell) \in W_3$$

where  $(\cdot)$  denotes the quantities to be summed, i.e.  $\{a(m,n)r(k-m, \ell-n)\}$ .

Now taking Frobenius ( $\|\cdot\|_F$ ) norms on both sides of the above equation, using (A-8), the bounds in (4) for  $r(k,\ell)$  and  $a(k,\ell)$ , and evaluating the resulting summations, we find that for some positive constants  $C_1, C_2, C_3$ .

$$\|\underline{a}'_{p,q} - \underline{a}\|_F \leq \|R_{p,q}^{-1}\|_2 (C_1 \alpha_1^p + C_2 \alpha_2^q + C_3 \alpha_1^p \alpha_2^q)$$

Using (A-6),  $\|R_{p,q}^{-1}\|_2 = 1/\lambda_{\min}(R_{p,q})$ . Also, since  $\lambda_{\min}(R_{p,q})$  is monotonically non-increasing with increasing  $(p,q)$  [see (A-9)] with limiting value  $S = \inf S(\omega_1, \omega_2)$ , the above bound may be written as

$$\sum_{(m,n) \in W_3} [a'_{p,q}(m,n) - a(m,n)]^2 \leq \frac{1}{S} (C_1 \alpha_1^p + C_2 \alpha_2^q + C_3 \alpha_1^p \alpha_2^q)$$

$$\text{Hence, } \lim_{p,q \rightarrow \infty} \sum_{m,n \in W_3} [a'_{p,q}(m,n) - a(m,n)]^2 = 0$$

This implies that

$$\lim_{p,q \rightarrow \infty} \sum_{m,n \in W_3} |a'_{p,q}(m,n) - a(m,n)| = 0 \quad (\text{A-16})$$

i.e. the sequence  $\{a'_{p,q}(m,n) - a(m,n)\}$ , indexed in  $(p,q)$ , with  $(m,n) \in W_3$  is absolutely summable on the unit circles,  $|z_1| = |z_2| = 1$  and converges to zero as  $(p,q) \rightarrow \infty$

Now

$$|A'_{p,q}(z_1, z_2) - A(z_1, z_2)| = \left| \sum_{(k,l) \in W_3} [a'_{p,q}(k,l) - a(k,l)] z_1^{-k} z_2^{-l} - \sum_{(k,l) \notin W_3} a(k,l) z_1^{-k} z_2^{-l} \right|$$

Using the bound for  $a(k,l)$  and evaluating the second summation, we get for some positive constants  $K_1, K_2$ ,

$$|A'_{p,q}(z_1, z_2) - A(z_1, z_2)| \leq \sum_{k,l \in W_3} |a'_{p,q}(k,l) - a(k,l)| + K_1 \alpha_1^p + K_2 \alpha_2^q ;$$

$$|z_1| = |z_2| = 1$$

$$\text{Letting } \epsilon(p,q) = \sum_{k,l \in W_3} |a'_{p,q}(k,l) - a(k,l)| + K_1 \alpha_1^p + K_2 \alpha_2^q$$

we have

$$|A'_{p,q}(z_1, z_2) - A(z_1, z_2)| < \epsilon(p, q) \quad , \quad |z_1| = |z_2| = 1 \quad (A-17)$$

Since  $\lim_{p,q \rightarrow \infty} \epsilon(p, q) = 0$  it follows that  $\lim_{(p,q) \rightarrow \infty} A'_{p,q}(z_1, z_2) = A(z_1, z_2)$  uniformly on  $|z_1| = |z_2| = 1$ .

Now,  $\{\beta_{p,q}^2\}$  is non-increasing with  $(p, q)$  with lower limit  $\beta^2$  given by (46), since by assumption  $S^{-1}(\omega_1, \omega_2)$  exists. Thus, similar to the 1-D non-causal case,

$$|A_{p,q}(z_1, z_2) - A(z_1, z_2)| < \epsilon'(p, q) \quad , \quad |z_1| = |z_2| = 1$$

This establishes the uniform convergence of the PEFs to the limit PEF on the unit circles.

b) Stability:  $A(z_1, z_2)$  is PA on the unit circles. Let  $a = \inf A(\omega_1, \omega_2) > 0$ .

Since  $|A_{p,q}(\omega_1, \omega_2) - A(\omega_1, \omega_2)| < \epsilon'(p, q)$ , we can find some  $(p_0, q_0) < \infty$  such that for  $(p, q) > (p_0, q_0)$ ,  $\epsilon'(p, q) < a$  implying that  $A(\omega_1, \omega_2) > a - \epsilon'(p, q) > 0$  ,  $(p, q) > (p_0, q_0)$

Hence, the non-causal models are guaranteed to be stable for

$$(p, q) > (p_0, q_0).$$

c) Correlation Match:

For  $(p, q) > (p_0, q_0)$ ,

$$|S(\omega_1, \omega_2) - S_{p,q}(\omega_1, \omega_2)| = \left| \frac{\beta^2}{A(\omega_1, \omega_2)} - \frac{\beta_{p,q}^2}{A_{p,q}(\omega_1, \omega_2)} \right| < k \epsilon(p, q) .$$

The above bound is obtained similar to the 1-D non-causal case. Hence, it follows that

$$|r(k, \ell) - r_{p,q}(k, \ell)| < k \epsilon(p, q) \quad , \quad (p, q) > (p_0, q_0) .$$

Proof of Theorem 3:

Proof: Since  $S(z_1, z_2)$  is PA,  $\tilde{S}(z_1, z_2) = \log S(z_1, z_2)$  is also analytic in a neighborhood of the unit circles, say,  $Z_1 = \{\alpha_1 < |z_1| < 1/\alpha_1, \alpha_2 < |z_2| < 1/\alpha_2; 0 < \alpha_1, \alpha_2 < 1\}$ . A unique Laurent expansion for  $\tilde{S}$  in  $Z_1$  can be written as

$$\tilde{S}(z_1, z_2) = \sum_{m,n} c(m,n) z_1^{-m} z_2^{-n}, \quad z_1, z_2 \in Z_1$$

The above series can be decomposed as  $\tilde{S} = \tilde{H}^+ + \tilde{H}^- + \tilde{S}_\epsilon$  [10] where

$$\tilde{H}^+ = \sum_{m=-\infty}^{\infty} c(m,0) z_1^{-m} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} c(m,n) z_1^{-m} z_2^{-n}$$

$$\tilde{H}^- = \sum_{m=-\infty}^{\infty} c(m,0) z_1^{-m} + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{-1} c(m,n) z_1^{-m} z_2^{-n}$$

$$\tilde{S}_\epsilon = - \sum_{m=-\infty}^{\infty} c(m,0) z_1^{-m}$$

and are analytic in  $Z_2 = \{\alpha_1 < |z_1| < 1/\alpha_1, |z_2| > \alpha_2\}$ ,  $Z_3 = \{\alpha_1 < |z_1| < 1/\alpha_1, |z_2| < \alpha_2\}$ , and  $Z_4 = \{\alpha_1 < |z_1| < 1/\alpha_1\}$ , respectively. The positive monotone property of the exponential function implies that  $e^{\pm \tilde{H}^+}$ ,  $e^{\pm \tilde{H}^-}$ , and  $e^{\tilde{S}_\epsilon}$  are also analytic in  $Z_2$ ,  $Z_3$ , and  $Z_4$ , respectively.

Hence, we may write

$$S = \frac{e^{\tilde{S}_\epsilon}}{e^{-\tilde{H}^+} e^{-\tilde{H}^-}}$$

Defining  $A'(z_1, z_2) \triangleq e^{-\tilde{H}^+}$ , we have

$$S(z_1, z_2) = a'_0(z_1) / A'(z_1, z_2) A'(z_1^{-1}, z_2^{-1})$$

where  $a'_0(z_1) = e^{\tilde{S}_\epsilon} = A'(z_1, \infty)$ , is a 1-D SDF.

The arguments above show that  $A'(z_1, z_2)$  is semi-minimum phase. With

$$A(z_1, z_2) \triangleq 1 - \sum_{m, n \in \hat{S}_2} a(m, n) z_1^{-m} z_2^{-n} = \beta^2 A'(z_1, z_2)$$

we get

$$S(z_1, z_2) = \frac{\beta^2 a_0(z_1)}{A(z_1, z_2) A(z_1^{-1}, z_2^{-1})}$$

With the given cepstral decomposition, and the requirement that  $a(0,0)=1$ , it follows that the decomposition is unique.

Proof of Theorem 4:

The proof of this theorem is similar to that of Marzetta's [7,8] for  $\hat{A}_n(z_1, z_2)$  from which the contention for  $A_n(z_1, z_2)$  follows. The convergence of the SDFs can be shown similar to Theorem 1 where all the quantities are now functions of  $z_1$ . The detailed proof may be found in [16].

Proof of Theorem 5:

- a) With the PA assumption on the SDF,  $S(z_1, z_2)$  and following the method of proof in Theorem 2, it is easy to prove that

$$\sum_{(m,n) \in W_2} [a'_{p,q}(m,n) - a(m,n)]^2 < C_1 \alpha_1^p + C_2 \alpha_2^q + C_3 \alpha_1^p \alpha_2^q, \quad 0 < \alpha_1, \alpha_2 < 1$$

where

$$a'_{p,q}(m,n) \triangleq \frac{\beta^2}{\beta_{p,q}^2} a_{p,q}(m,n), \quad (m,n) \in W_2 \quad (A-18)$$

Hence,

$$\lim_{p,q \rightarrow \infty} \sum_{(m,n) \in W_2} |a'_{p,q}(m,n) - a(m,n)| = 0 \quad (A-19)$$

implying that the sequence  $\{a'_{p,q}(m,n) - a(m,n)\}$ , indexed in  $(p,q)$  with  $(m,n) \in W_2$  is absolutely summable on the unit circles, and

converges to zero as  $(p,q) \rightarrow \infty$ .

Now, for  $Z_1 = \{|z_1| = 1, |z_2| \geq 1\}$ ,

$$\begin{aligned} |A'_{p,q}(z_1, z_2) - A(z_1, z_2)| &\leq \sum_{(m,n) \in W_2} |a'_{p,q}(m,n) - a(m,n)| |z_1|^{-m} |z_2|^{-n} \\ &+ \sum_{(m,n) \notin W_2} |a(m,n)| |z_1|^{-m} |z_2|^{-n} \leq \sum_{(m,n) \in W_2} |a'_{p,q}(m,n) - a(m,n)| \\ &+ \sum_{(m,n) \notin W_2} |a(m,n)| \triangleq \varepsilon(p,q). \end{aligned} \quad (A-20)$$

The PA assumption for  $S(z_1, z_2)$  implies that  $\{a(m,n)\}$  are exponentially bounded. This and (A-18) together imply that  $\lim_{p,q \rightarrow \infty} \varepsilon(p,q) \rightarrow 0$ .

Hence,  $\lim_{p,q \rightarrow \infty} A'_{p,q}(z_1, z_2) \rightarrow A(z_1, z_2)$  uniformly on  $(z_1, z_2) \in Z_1$ .

Similar to non-causal models, it can be established that  $\{\beta_{p,q}^2\}$  is a nonincreasing sequence that converges to a positive lower limit,  $\beta^2$ .

We now give an expression for  $\beta^2$ , which is also given in [10].

Defining  $d(z_1) \triangleq \beta^2 a_0^{-1}(z_1)$ , and from Theorems 3 and 4, it follows that  $d(z_1)$  is PA in a neighborhood of  $|z_1| = 1$ , so that  $d^{-1}(z_1)$  is likewise.

$$\text{Now, } \beta^2 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S_{\varepsilon}(\omega_1, \omega_2) d\omega_1 d\omega_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta^2 a_0(\omega_1) d\omega_1$$

Thus,

$$\beta^2 = 1 / \frac{1}{2\pi} \int_{-\pi}^{\pi} d^{-1}(\omega_1) d\omega_1 > 0$$

Using (A18-A20), we obtain

$$|A'_{p,q}(z_1, z_2) - A(z_1, z_2)| < [\beta_{p,q}^2 \varepsilon(p,q) + (\beta_{p,q}^2 - \beta^2) A_{\max}] / \beta^2 \triangleq \varepsilon'(p,q), \quad (A-21)$$

$$z_1, z_2 \in Z_1$$

where  $A_{\max} \triangleq \sup |A(z_1, z_2)|$ ,  $z_1, z_2 \in Z_1$ .

This proves the uniform convergence of the sequence of the semi-causal PEFs  $A_{p,q}(z_1, z_2)$  to the analytic, semi-causal limit PEF  $A(z_1, z_2)$  in  $Z_1$ .

b) To prove stability, we note from (A-21) that

$$|r_{p,q} - r| < \epsilon'(p, q)$$

$$\text{and} \quad |i_{p,q} - i| < \epsilon'(p, q)$$

where  $r_{p,q}$ ,  $r$  and  $i_{p,q}$ ,  $i$  represent the real and imaginary parts of  $A_{p,q}(z_1, z_2)$  and  $A(z_1, z_2)$  for any  $z_1, z_2 \in Z_1$ . Since  $r$  and  $i$  are not simultaneously zero, we can find some  $(p_0, q_0) < \infty$  such that for  $(p, q) > (p_0, q_0)$ ,  $r_{p,q}$  and  $i_{p,q}$  are sufficiently close to  $r$  and  $i$ , respectively. This implies that the models will be stable for  $(p, q) > (p_0, q_0)$ .

c) We now prove uniform convergence of the spectra realized by the models for  $(p, q) > (p_0, q_0)$  to the given SDF on the unit circles. This in turn implies increasingly accurate correlation match. Since for  $(p, q) > (p_0, q_0)$ ,  $A_{p,q}(z_1, z_2)$  has no zeroes in  $Z_1$ , it is semi-minimum phase, and it is easy to establish the uniform convergence of  $1/A_{p,q}(z_1, z_2)$  to  $1/A(z_1, z_2)$ . Also, since  $a_{0p,q}(z_1) = A_{p,q}(z_1, \infty)$ , it follows that  $a_{0p,q}(z_1)$  converges uniformly to  $a_0(z_1)$  on  $|z_1| = 1$ . Since  $\beta_{p,q}^2 \rightarrow \beta^2$  monotonically, and since the PEF sequence can be uniformly bounded, it follows that

$$\frac{\beta_{p,q}^2 a_{0p,q}(z_1)}{A_{p,q}(z_1, z_2) A_{p,q}(z_1^{-1}, z_2^{-1})} \rightarrow \frac{\beta^2 a_0(z_1)}{A(z_1, z_2) A(z_1^{-1}, z_2^{-1})} \quad |z_1| = |z_2| = 1$$

uniformly, i.e.

$$|S_{p,q}(z_1, z_2) - S(z_1, z_2)| < \epsilon''(p, q) \quad |z_1| = |z_2| = 1$$

In proving the uniform convergence of the SDFs from that of the PEF



sequence, we obtain a bound  $\epsilon''(p,q)$  which may be weaker than  $\epsilon'(p,q)$ .

However,  $\lim_{(p,q) \rightarrow \infty} \epsilon''(p,q) = 0$ .

If  $\{r_{p,q}(k,\ell)\}$  are the covariances realized by the  $(p,q)$  order model, then,

$$|r_{p,q}(k,\ell) - r(k,\ell)| \leq \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |S_{p,q}(\omega_1, \omega_2) - S(\omega_1, \omega_2)| d\omega_1 d\omega_2 = \epsilon''(p,q)$$

This proves that correlation match gets increasingly better with  $(p,q) > (p_0, q_0)$ .

# REFERENCES

1. W. K. Pratt, "Digital Image Processing," Wiley, N.Y., 1978.
2. A. K. Jain, "Image Data Compression - A Review," Proc. IEEE, Vol. 69, pp. 349-389, March 1981.
3. J. L. Doob, "Stochastic Processes," Wiley, N.Y., 1953.
4. J. Makhoul, "Linear Prediction: A Tutorial Review," Proc. IEEE, Vol. 63, pp. 561-580, April 1975.
5. P. Whittle, "On Stationary Processes in the Plane," Biometrika, Vol. 41, pp. 434-449, Dec. 1954.
6. M. P. Ekstrom and J. W. Woods, "Two-Dimensional Spectral Factorization with Applications in Recursive Digital Filtering," IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-24, #2, pp. 115-128, April 1976.
7. T. L. Marzetta, "Two-Dimensional Linear Prediction: Arrays Minimum Phase Prediction Error Filters, and Reflection Coefficient Arrays," IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-28, #6, pp. 725-733, Dec. 1980.
8. T. L. Marzetta, "A Linear Prediction Approach to Two-Dimensional Spectral Factorization and Spectral Estimation," Ph.D. Dissertation, Dept. Elec. Engg. & Comput. Sc.; MIT, Cambridge, MA, Feb. 1978.
9. A. K. Jain, "Partial Differential Equations and Finite Difference Methods in Image Processing, Part I - Image Representation," J. Opt. Theo. Appl., Vol. 23, #1, pp. 65-91, Sept. 1977.
10. A. K. Jain, "Advances in Mathematical Models for Image Processing," Proc. IEEE, Vol. 69, #5, pp. 502-528, May 1981.
11. A. Oppenheim and R. Schaffer, "Digital Signal Processing," Prentice-Hall, 1975.
12. R. Bellman, "Introduction to Matrix Analysis," McGraw Hill, New York, 1970.
13. A. K. Jain, "A Fast Karhunen-Loeve Transform for a Class of Random Processes," IEEE Trans. Commun., Vol. COM-24, pp. 1023-1029.
14. A. K. Jain, "A Fast Karhunen-Loeve Transform for Recursive Filtering of Images Corrupted by White and Colored Noise," IEEE Trans. Comput., Vol. C-26, pp. 560-571, June 1977.
15. J. W. Woods, "Two-Dimensional Discrete Markovian Fields," IEEE Trans. Inform. Theory, Vol. IT-18, pp. 232-240, March 1972.
16. S. Ranganath, "Two-Dimensional Spectral Factorization, Spectral Estimation, and Applications in Image Processing," Ph.D. Dissertation, Dept. Elec. Computer Engg., University of California, Davis, March 1983.

## LIST OF FIGURES

- Figure 1: Prediction regions and typical prediction windows for causal, semi-causal, and non-causal geometries.
- Figure 2: An example of semi-causal filter geometry obtained by the rationalization procedure.
- a) Reflection coefficient geometry
  - b) Geometry for  $\hat{a}_n(m)$
  - c) Prediction geometry for the semi-causal model
- Figure 3: A method of obtaining finite order semi-causal models via Levinson's recursions.
- Figure 4: Covariances corresponding to  $r(k,l) = .9^{\sqrt{k^2+l^2}}$  on the quarter plane.
- Figure 5: PEFs and covariance mismatches for  $(p,q) = (1,1)$  and  $(2,1)$  non-causal models.
- a) PEF coefficients corresponding to  $(p,q) = (1,1)$  non-causal model
  - b) PEF coefficients corresponding to  $(p,q) = (1,2)$  non-causal model
  - c)  $\{r(k,l) - r_{1,1}(k,l)\}$
  - d)  $\{r(k,l) - r_{1,2}(k,l)\}$
- Figure 6: Infinite order semi-causal PEF obtained by Wiener-Doob factorization.
- Figure 7: Rationalized semi-causal model corresponding to  $q=2$ ,  $\{\rho_1(k)\}$  truncated to 5 terms,  $\{\rho_2(k)\}$  truncated to 3 terms, and  $p=2$ .
- a) Coefficients of semi-causal PEF
  - b) Covariance mismatch,  $\{r(k,l) - \hat{r}(k,l)\}$
- Figure 8: Semi-causal model obtained using  $q=1$ , truncating  $\{\rho_1(k)\}$  to 15 terms, and  $p=2$ .
- a) Semi-causal PEF coefficients
  - b) Covariance mismatch,  $\{r(k,l) - \hat{r}(k,l)\}$
- Figure 9: Semi-causal models obtained by solving finite order linear equations.
- a) PEF coefficients of  $(p,q) = (1,1)$  semi-causal model
  - b) PEF coefficients of  $(p,q) = (2,2)$  semi-causal model

- c) PEF coefficients of  $(p,q) = (3,3)$  semi-causal model

Figure 10: Covariance mismatches of the semi-causal models in Figure 9.

- a)  $\{r(k,\ell) - r_{1,1}(k,\ell)\}$
- b)  $\{r(k,\ell) - r_{2,2}(k,\ell)\}$
- c)  $\{r(k,\ell) - r_{3,3}(k,\ell)\}$

Figure 11: Infinite order causal PEF obtained by Wiener-Doob Factorization

Figure 12: Rationalized causal model corresponding to  $q=2$ ,  $\{\rho_1(k)\}$  truncated to 5 terms,  $\{\rho_2(k)\}$  truncated to 3 terms, and  $p=2$ .

- a) Coefficients of causal PEF
- b) Covariance mismatch,  $\{r(k,\ell) - \hat{r}(k,\ell)\}$

Figure 13: Rationalized causal model obtained using  $q=1$ ,  $\{\rho_1(k)\}$  truncated to 15 terms, and  $p=5$ .

- a) Coefficients of causal PEF
- b) Covariance mismatch,  $\{r(k,\ell) - \hat{r}(k,\ell)\}$

Figure 14: Causal models obtained by solving finite order linear equations.

- a) PEF coefficients of  $(p,q) = (1,1)$  causal model
- b) PEF coefficients of  $(p,q) = (2,2)$  causal model
- c) PEF coefficients of  $(p,q) = (3,3)$  causal model

Figure 15: Covariance mismatches for the causal models in Figure 14.

- a)  $\{r(k,\ell) - r_{1,1}(k,\ell)\}$
- b)  $\{r(k,\ell) - r_{2,2}(k,\ell)\}$
- c)  $\{r(k,\ell) - r_{3,3}(k,\ell)\}$

**DATE**  
**ILME**